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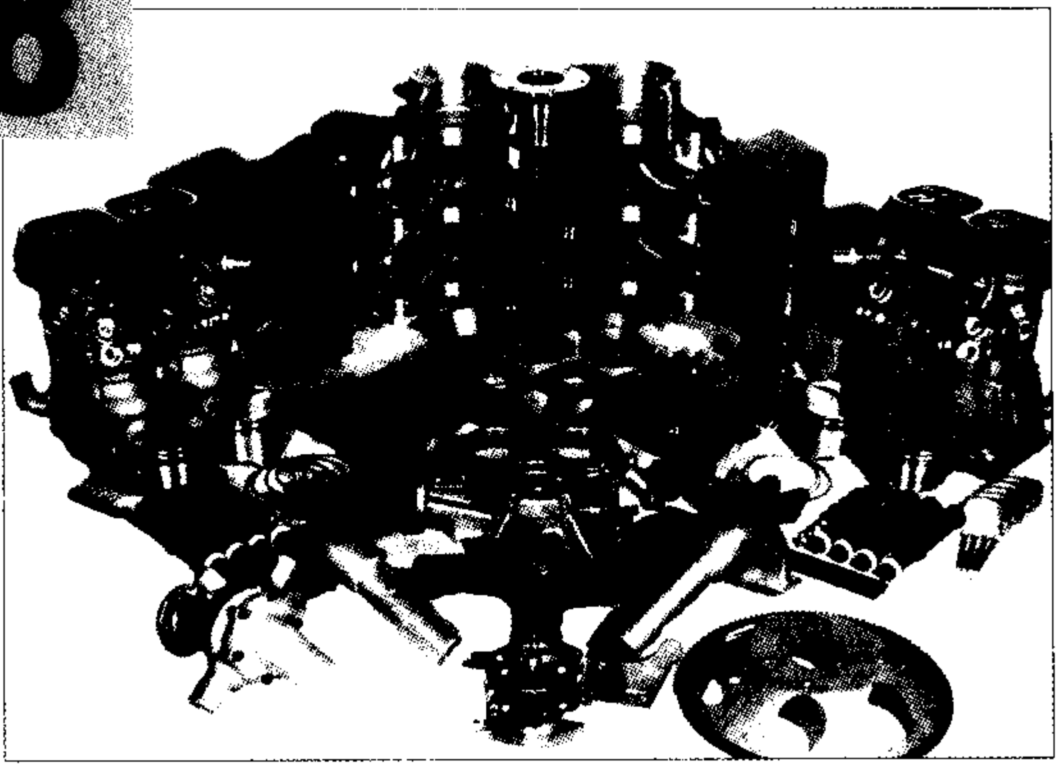


GOVERNORS

8.1 INTRODUCTION

A governor is a device to maintain, as closely as possible, a constant mean speed of rotation of the crankshaft over long periods during which the load on the engine may vary. The governor meets the varying demand for power by regulating the supply of working fluid.

8.2 TYPES OF GOVERNORS

The two types of governors are:

1. Centrifugal governors, and
2. Inertia governor.

In centrifugal governors, the centrifugal force is balanced by the controlling force. These type of governors are used extensively. In inertia type of governors, the inertia force is balanced by the controlling force. They are not used popularly.

The centrifugal governors can be further classified as follows:

1. Pendulum type—Watt governor.
2. Loaded type

(a) Dead weight type
  (i) Porter governor
  (ii) Proell governor
(b) Spring loaded type
  (i) Hartnell governor
  (ii) Hartung governor
  (iii) Wilson–Hartnell governor
  (iv) Pickering governor

8.3 CENTRIFUGAL GOVERNORS

8.3.1 Simple Watt Governor

A simple Watt governor is shown in Fig.8.1(a). It consists of two balls attached to the spindle through four arms. The upper two arms meet at the pivot, which may be on the spindle axis or offset from the spindle axis. The arms may be of the open type or crossed type. The lower arms are connected to a sleeve by pin joints. The movement of the sleeve is restricted by means of two stops.

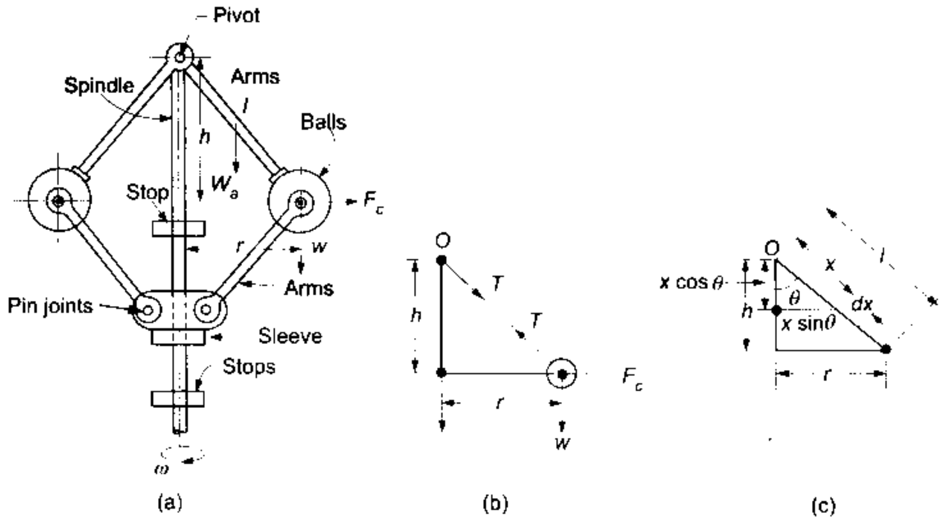


Fig.8.1 Simple Watt governor

Neglecting weight of the arms

Let w = weight of the balls
r = radius of the balls
h = height of the governor
T = tension in the arms
ω = angular speed of rotation

The forces acting on the ball are shown in Fig.8.1(b).

Centrifugal force due to the balls, $F_c = \left(\frac{w}{g}\right) \cdot \omega^2 r$

Taking moments about the pivot, we have

$$F_c h = wr$$

or $$(w/g)\cdot\omega^2 rh = wr$$

or $$h = \frac{g}{\omega^2}$$

or $$h = \frac{g}{\left(2\pi\frac{N}{60}\right)^2} = \frac{k}{N^2} \tag{8.1}$$

where $k = 894.565$ is a constant.

$$\frac{dh}{dN} = \frac{-2k}{N^3}$$

Change in height, $$\delta h = -\left(\frac{2k}{N^2}\right)\cdot\left(\frac{\delta N}{N}\right) \tag{8.2}$$

With increasing speed, δh becomes insignificant, and governor stops functioning. It is used for slow speed engines.

Considering weight of the arms

Let w_a = weight of the arm per unit length

W_a = total weight of the arm = $w_a l$

l = length of the arm

θ = angle subtended by the arm with the spindle axis

The forces acting on the ball are shown in Fig.8.1(c).

Consider an elementary length dx of the arm at a distance x from the pivot. Weight of the elementary length = $w_a \cdot dx$

Radius of the elementary length = $x\sin\theta$

Centrifugal force due to the elementary length, $$dF_a = \left(w_a\cdot\frac{dx}{g}\right)\cdot\omega^2\cdot x\sin\theta$$

Moment about the pivot = $dF_a \cdot x\cos\theta$

Total moment $$= \left(\frac{w_a}{g}\right)\omega^2\sin\theta\cos\theta\int_0^l x^2 dx = \left(\frac{w_a}{g}\right)\omega^2\sin\theta\cos\theta\left(\frac{l^3}{3}\right)$$

$$= \left(\frac{w_a l}{3g}\right)\omega^2\cdot l\sin\theta\cdot l\cos\theta = \left(\frac{1}{3}\right)\cdot\left(\frac{W_a}{g}\right)\omega^2 rh$$

Therefore, the effect of the weight of the arm is equivalent to that produced by a weight $\left(\frac{W_a}{3}\right)$ placed at the centre of the ball.

Taking moments about the pivot point O, we have

$$\left(\frac{W_a}{3g}\right)\omega^2 rh + \left(\frac{w}{g}\right)\omega^2 rh = \left(\frac{W_a}{2}\right)l\sin\theta + wr$$

$$\left[\frac{\left(w+\frac{W_a}{3}\right)}{g}\right]\omega^2 rh = \left(w+\frac{W_a}{2}\right)r$$

or
$$h = \left(\frac{g}{\omega^2}\right)\left[\frac{\left(w + \frac{W_a}{2}\right)}{\left(w + \frac{W_a}{3}\right)}\right] \tag{8.3}$$

Example 8.1

A simple Watt governor rotates at 75 rpm. Calculate its vertical height and the change if the speed increases to 80 rpm. Also calculate the height at 75 rpm if the weight of the ball is 20 N and that of the arm 5 N.

■ **Solution**

$$h_1 = \frac{g}{4\pi^2 N^2} = \frac{9.81 \times 3600}{4\pi^2 \times 75^2} = 0.159 \text{ m}$$

$$h_2 = 0.159\left(\frac{75}{80}\right)^2 = 0.14 \text{ m}$$

Change in height, $h_1 - h_2 = 0.159 - 0.14 = 0.019 \text{ m}$ or 19 mm

Height of the governor considering the weight of the arm,

$$h = \left(\frac{g}{w^2}\right)\left[\frac{w + \frac{W_a}{2}}{w + \frac{W_a}{3}}\right]$$

$$= \left(\frac{9.81 \times 3600}{4\pi^2 \times 75^2}\right)\left[\frac{20 + \frac{5}{2}}{20 + \frac{5}{3}}\right] = 0.165 \text{ m}$$

8.3.2 Gravity Loaded Type Governors

Porter governor The Porter governor is a modification of the Watt governor in which a central mass is attached to the sleeve. The Porter governor is shown in Fig.8.2(a). The forces acting on the governor are shown in Fig.8.2(b).

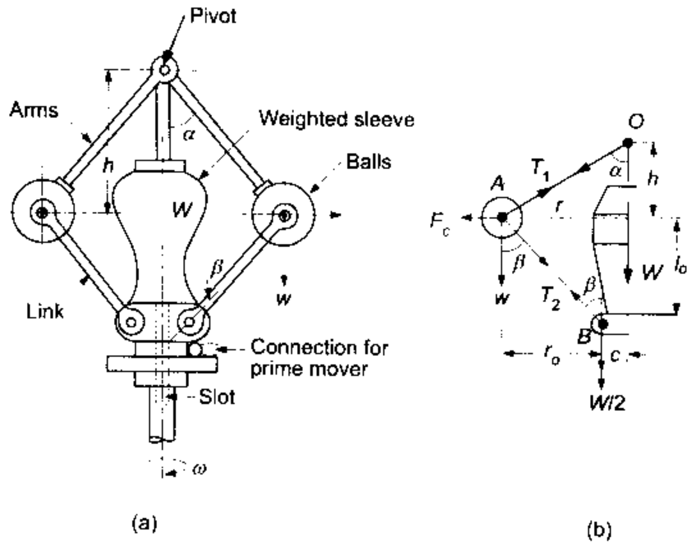


Fig.8.2 Porter governor

Let W = dead weight of sleeve
w = weight of ball
T_1 = tension in upper arm
T_2 = tension in lower arm
r = radius of the balls
$r_o = r - c$
c = distance of hinge B from axis of rotation
ω = angular speed of rotation

The forces acting at the hinge B are:

1. Half of the central load W.
2. Tension T_2 in the lower arm.
3. Reaction of the hinge.

Resolving the forces vertically, we have

$$T_2 \cos\beta = \frac{W}{2}$$

or

$$T_2 = \frac{W}{2\cos\beta}$$

The ball is in equilibrium under the following forces:

1. Centrifugal force, F_c.
2. Weight of the ball, w.
3. Tension in upper arm, T_1.

Resolving the forces horizontally, we have

$$\begin{aligned} F_c &= T_1 \sin\alpha + T_2 \sin\beta \\ &= T_1 \sin\alpha + \left(\frac{W}{2}\right)\tan\beta \end{aligned}$$

Resolving the forces vertically, we have

$$T_1 \cos\alpha = w + T_2 \cos\beta = w + \frac{W}{2}$$

Therefore,

$$\begin{aligned} F_c &= \left(\frac{W}{2}\right)\tan\beta + \left(w + \frac{W}{2}\right)\tan\alpha \\ &= \left(\frac{w}{g}\right)\omega^2 r \end{aligned}$$

If $\tan\alpha = \frac{r}{h}$ and $\tan\beta = \frac{r_o}{l}$, then

$$\begin{aligned} \left(\frac{w}{g}\right)\omega^2 r &= \left(\frac{W}{2}\right)\tan\beta + \left(w + \frac{W}{2}\right)\tan\alpha \\ &= \left[\left(w + \frac{W}{2}\right) + \left(\frac{W}{2}\right)\left(\frac{\tan\beta}{\tan\alpha}\right)\right]\left(\frac{r}{h}\right) \end{aligned}$$

$$= \left[w + \left(\frac{W}{2}\right)(1+k)\right]\left(\frac{r}{h}\right)$$

or
$$h = \left(\frac{g}{\omega^2}\right)\left[1 + \left(\frac{W}{2w}\right)(1+k)\right] \tag{8.4}$$

where $k = \tan\beta/\tan\alpha$.

If $\alpha = \beta$, then

$$\left(\frac{w}{g}\right)\omega^2 r = (w + W)\cdot\left(\frac{r}{h}\right)$$

$$h = \left(\frac{g}{\omega^2}\right)\left[1 + \frac{W}{w}\right] \tag{8.5}$$

If F is the frictional force acting on the sleeve, then

$$h = \left(\frac{g}{\omega^2}\right)\left[\frac{w + W \pm F}{w}\right] \tag{8.6}$$

Take the +ve sign when the sleeve moves upwards or the governor speed increases and the −ve sign when the sleeve moves downwards or the governor speed decreases.

Example 8.2

The arms of a Porter governor are each 200 mm long. The weight of each ball is 40 N and that of the sleeve is 200 N. The radius of rotation of the balls is 125 mm when the sleeve begins to rise and reaches a value of 150 mm for maximum speed. Determine the speed range of the governor. If the friction at the sleeve is equivalent to 20 N of load at the sleeve, determine how the speed range is modified.

■ **Solution**

$$h_1 = \left[200^2 - 125^2\right]^{0.5} = 156.12 \text{ mm}$$

$$\omega_1^2 = \left(1 + \frac{W}{w}\right)\left(\frac{g}{h_1}\right) = \left(1 + \frac{200}{40}\right)\left(\frac{9.81}{156.12 \times 10^{-3}}\right) = 378.02$$

$$\omega_1 = 19.417 \text{ rad/s}$$

$$N_1 = 185.4 \text{rpm}$$

$$h_2 = \left[200^2 - 150^2\right]^{0.5} = 132.28 \text{ mm}$$

$$\omega_2^2 = \left(1 + \frac{200}{40}\right)\left(\frac{9.81}{132.28 \times 10^{-3}}\right) = 444.96$$

$$\omega_2 = 21.09 \text{ rad/s}$$

$$N_2 = 201.4 \text{ rpm}$$

Range of speed $\quad N_2 - N_1 = 201.4 - 185.4 = 16$ rpm

When the sleeve moves downwards, the force of friction acts upwards, therefore

$$\omega_1^2 = \left[\frac{w + W - F}{w}\right]\left(\frac{g}{h_1}\right)$$

$$= \left[\frac{40 + 200 - 20}{40}\right]\left(\frac{9.81}{156.12 \times 10^{-3}}\right) = 345.6$$

$$\omega_1 = 18.59$$

$$N_1 = 177.52 \text{ rpm}$$

When the sleeve moves upwards, the force of friction acts downwards, therefore

$$\omega_2^2 = \left[\frac{w + W + F}{w}\right]\left(\frac{g}{h_2}\right)$$

$$= \left[\frac{40 + 200 + 20}{40}\right]\left(\frac{9.81}{132.28 \times 10^{-3}}\right) = 482.04$$

$$\omega_2 = 21.96 \text{ rad/s}$$

$$N_2 = 209.66 \text{ rpm}$$

Range of speed, $N_2 - N_1 = 209.66 - 177.52 = 32.14$ rpm

Example 8.3

The arms of a Porter governor are 250 mm long. The upper arms are pivoted on the axis of revolution, but the lower arms are attached to a sleeve at a distance of 50 mm from the axis of rotation. The weight on the sleeve is 600 N and the weight of each ball is 80 N. Determine the equilibrium speed when the radius of rotation of the balls is 150 mm. If the friction is equivalent to a load of 25 N at the sleeve, determine the range of speed for this position.

■ **Solution**

$$h = \left[250^2 - 150^2\right]^{0.5} = 200 \text{ mm} = 0.2 \text{ m}$$

$$r_o = r - c = 150 - 50 = 100 \text{ mm}$$

$$l = \left[250^2 - 100^2\right]^{0.5} = 229.13 \text{ mm} = 0.22913 \text{ m}$$

$$\tan\alpha = \frac{r}{h} = \frac{150}{200} = 0.75$$

$$\tan\beta = \frac{r_o}{l} = \frac{100}{229.13} = 0.43643$$

$$k = \frac{\tan\beta}{\tan\alpha} = \frac{0.43643}{0.75} = 0.582$$

$$\omega^2 = \left[1 + \left(\frac{W}{2w}\right)(1 + k)\right]\left(\frac{g}{h}\right)$$

$$= \left[1 + \left(\frac{600}{160}\right)(1 + 0.582)\right]\left(\frac{9.81}{0.2}\right) = 340.04$$

$$\omega = 18.44 \text{ rad/s}$$

$$N = 176 \text{ rpm}$$

Maximum equilibrium speed occurs when the sleeve is going upwards.

$$\omega_2^2 = \left[1 + \left(\frac{(W + F)}{2w}\right)(1 + k)\right]\left(\frac{g}{h}\right)$$

$$= \left[1 + \left(\frac{600 + 25}{160}\right)(1 + 0.582)\right]\left(\frac{9.81}{0.2}\right) = 352.16$$

$$\omega_{\max} = 18.766 \text{ rad/s}$$

$$N_{\max} = 179.2 \text{ rpm}$$

$$\omega_1^2 = \left[1 + \left(\frac{W - F}{2w}\right)(1 + k)\right]\left(\frac{g}{h}\right)$$

$$= \left[1 + \left(\frac{600 - 25}{160}\right)(1 + 0.582)\right]\left(\frac{9.81}{0.2}\right) = 327.91$$

$$\omega_{min} = 18.11 \text{ rad/s}$$

$$N_{min} = 172.9 \text{ rpm}$$

$$\text{Range of speed} = 179.2 - 172.9 = 6.3 \text{ rpm}$$

Proell governor The Proell governor is shown in Fig.8.3(a), in which the balls are fixed at *C* and *D* to the extension of links *EB* and *FA*. The forces acting on the governor are shown in Fig.8.3(b).

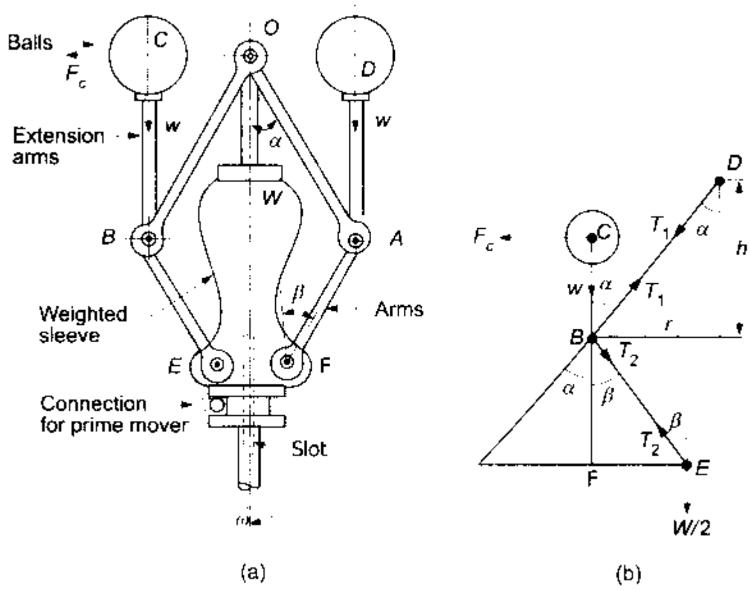


Fig.8.3 Proell governor

Considering the equilibrium of forces at point *E*, we have

$$T_2 \cos\beta = \frac{W}{2}$$

or
$$T_2 = \frac{W}{2\cos\beta}$$

At point *B*, we have

$$T_1 \cos\alpha = T_2 \cos\beta + w = \frac{W}{2} + w$$

Taking moments about *E*, we have

$$F_c \times CF + w \times FE - T_1 \cos\alpha \times FE - T_1 \sin\alpha \times BF = 0$$

$$F_c \times CF + w \times BF\tan\beta - \left(w + \frac{W}{2}\right) \times BF\tan\beta - \left(w + \frac{W}{2}\right) \times BF\tan\alpha = 0$$

$$F_c = \left(\frac{BF}{CF}\right)\left[\left(w + \frac{W}{2}\right)\tan\alpha + \left(\frac{W}{2}\right)\tan\beta\right]$$

Now $\tan\alpha = \dfrac{r}{h}$, and let $k = \dfrac{\tan\beta}{\tan\alpha}$.

$$\left(\frac{w}{g}\right)\omega^2 r = \left(\frac{BF}{CF}\right)\left[\left(\frac{W}{2}\right)(1+k)+w\right]\cdot\left(\frac{r}{h}\right)$$

$$\omega^2 = \left(\frac{BF}{CF}\right)\left[1+\left(\frac{W}{2w}\right)(1+k)\right]\cdot\left(\frac{g}{h}\right) \tag{8.7}$$

If $\alpha = \beta$, i.e. $k = 1$, then

$$\omega^2 = \left(\frac{BF}{CF}\right)\left[1+\frac{W}{w}\right]\left(\frac{g}{h}\right) \tag{8.8}$$

Example 8.4

The arms of a Proell governor are 300 mm long. The pivots of the upper and lower arms are 30 mm from the axis. The load on the sleeve is 250 N and the weight of each ball is 30 N. When the governor sleeve is at mid-position, the extension link of the lower arm is vertical and the radius of rotation of the balls is 160 mm. The vertical height of the governor is 200 mm.

If the speed of the governor is 150 rpm when at mid-position, find (a) length of the extension link, and (b) tension in the upper arm.

■ **Solution**

(a) Let BC = length of the extension link

At mid-position,

$$\alpha = \beta, \text{ or } k = 1$$

$$\omega^2 = \left(\frac{BF}{CF}\right)\left[1+\frac{W}{w}\right]\left(\frac{g}{h}\right)$$

$$BF = \left[300^2 - 130^2\right]^{0.5} = 270.37 \text{ mm}$$

$$\left(\frac{2\pi\times150}{60}\right)^2 = \left(\frac{270.37}{CF}\right)\left[1+\frac{250}{30}\right]\left(\frac{9.81}{0.2}\right)$$

$$246.74 = \frac{123775.4}{CF}$$

or

$$CF = 501.64 \text{ mm}$$

$$BC = CF - BF = 501.64 - 270.37 = 231.27 \text{ mm}$$

(b)

$$\cos\alpha = \frac{200}{300} = 0.667$$

$$T_1\cos\alpha = w + \frac{W}{2} = 30 + \frac{250}{2} = 155$$

$$T_1 = 232.5 \text{ N}$$

8.3.3 Spring Loaded Governors

Hartnell governor The Hartnell governor is of the spring loaded type, and is shown in Fig.8.4(a). It consists of two bell crank levers pivoted at points A to the frame. The frame is attached to the governor spindle and rotates with it. Each lever carries a ball at the end of the vertical arm AB and a roller at the other end of the horizontal arm. A helical compression spring provides equal downward forces on the two rollers through the sleeve. The spring force may be adjusted by the nut.

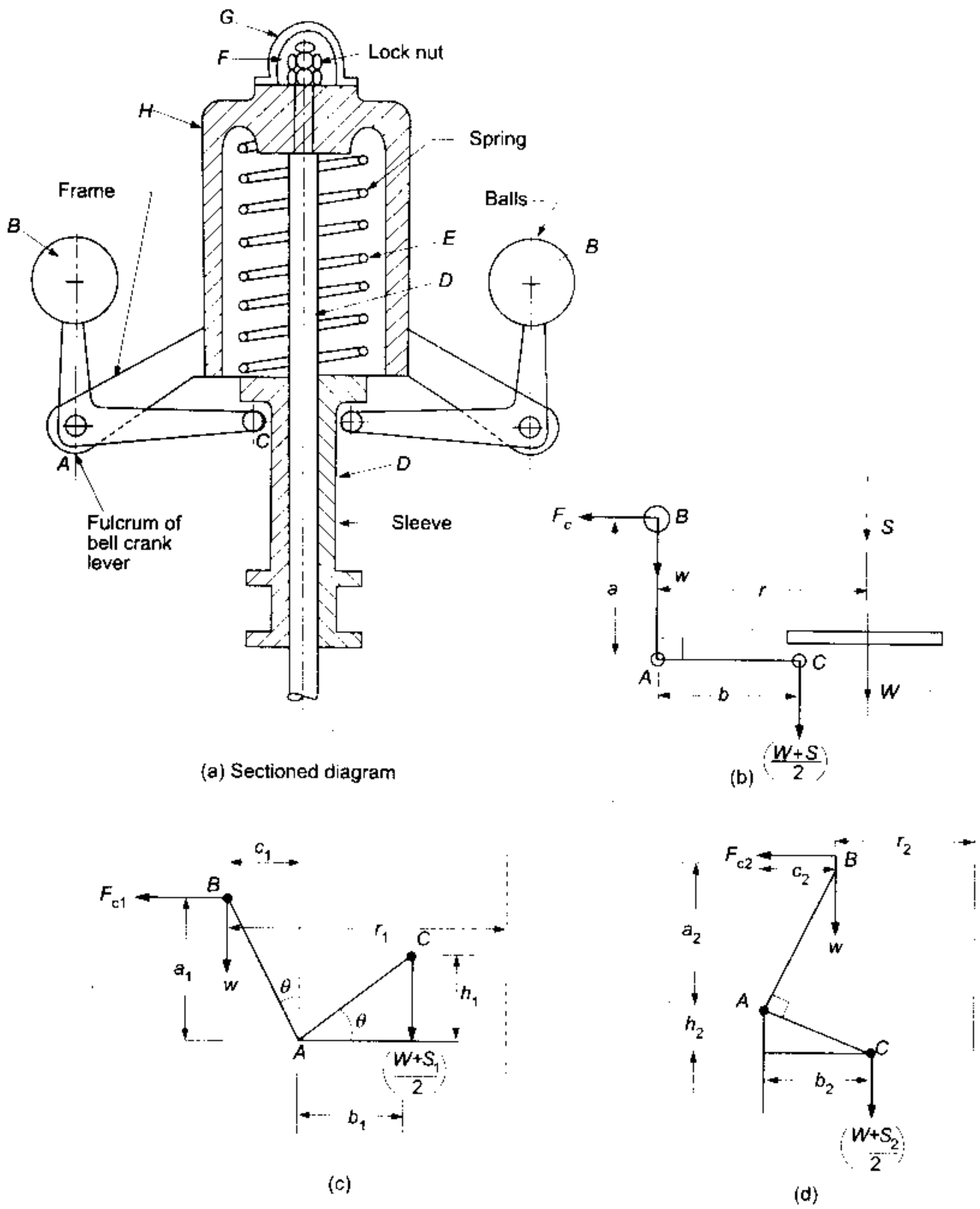


Fig.8.4 Hartnell governor

Let w = weight of the ball

W = weight of the sleeve

r_1, r_2 = maximum and minimum radii of rotation of the ball

ω_1, ω_2 = maximum and minimum angular speeds of rotation

S_1, S_2 = maximum and minimum spring forces

F_{c1}, F_{c2} = centrifugal forces at speeds ω_1 and ω_2 respectively

k = stiffness of the spring

a, b = vertical and horizontal length of arms of bell crank lever

r = radius of the ball

The forces acting on half of the governor are shown in Fig.8.4(b). The forces acting on half of the governor at maximum and minimum speeds are shown in Figs.8.4(c) and (d), respectively.

From Fig.8.4(b), taking moments about A, we have

$$F_c \times a = (W + S)\frac{b}{2}$$

$$W + S = \frac{2F_c a}{b} \tag{8.9}$$

From Fig.8.4(c), taking moments about A, we have

$$F_{c1}a_1 + wc_1 = (W + S_1)\frac{b_1}{2} \tag{8.10}$$

From Fig.8.4(d), taking moments about A, we have

$$F_{c2}a_2 - wc_2 = (W + S_2)\frac{b_2}{2} \tag{8.11}$$

Now $$c_1 + c_2 = r_1 - r_2$$

Neglecting the obliquity effect of arms, we have

$$a_1 = a_2 = a \quad \text{and} \quad b_1 = b_2 = b$$

$$W + S_1 = 2F_{c1}\frac{a}{b}$$

and $$W + S_2 = 2F_{c2}\frac{a}{b}$$

$$S_1 - S_2 = \frac{2a(F_{c1} - F_{c2})}{b} \tag{8.12}$$

Now $$\sin\theta = \frac{c_1}{a} = \frac{h_1}{b}$$

or $$h_1 = \frac{c_1 b}{a}$$

Similarly $$h_2 = \frac{c_2 b}{a}$$

$$h_1 + h_2 = \frac{b(c_1 + c_2)}{a}$$

Lift of sleeve, $$h = h_1 + h_2 = \frac{b(r_1 - r_2)}{a}$$

$$S_1 - S_2 = kh = \frac{kb(r_1 - r_2)}{a} \tag{8.13}$$

Comparing (8.12) and (8.13), we get

$$k = 2\left(\frac{a}{b}\right)^2\left[\frac{F_{c1} - F_{c2}}{r_1 - r_2}\right] \tag{8.14}$$

Example 8.5

A Hartnell governor moves between 300 rpm and 320 rpm for a sleeve lift of 20 mm. The sleeve arms and the ball arms are 80 mm and 120 mm, respectively. The levers are pivoted at 120 mm from the governor axis. The weight of each ball is 25 N. The ball arms are parallel to the governor axis at the lowest equilibrium speed. Determine (a) the loads on the spring at the minimum and the maximum speeds, and (b) the stiffness of the spring.

■ Solution

(a) Here $N_{max} = 320\,\text{rpm}$, $N_{min} = 300\,\text{rpm}$, $h = 20\,\text{mm}$, $a = 120\,\text{mm}$, $b = 80\,\text{mm}$, $w = 25\,\text{N}$, $r = 120\,\text{mm}$.

At the minimum equilibrium speed $N_{min} = 300$ rpm, the ball arms are parallel to the governor axis. Therefore $r = r_2 = 120$ mm.

$$F_{c2} = \left(\frac{w}{g}\right)\omega_{min}^2 r_2$$

$$= \left(\frac{25}{9.81}\right)\left(2\pi \times \frac{300}{60}\right)^2 \times 0.12 = 301.82\ \text{N}$$

Radius at the maximum speed,

$$r_1 = r_2 + h\left(\frac{a}{b}\right)$$

$$= 120 + 20\left(\frac{120}{80}\right) = 150\ \text{mm or } 0.15\ \text{m}$$

$$F_{c1} = \left(\frac{25}{9.81}\right)\left(2\pi \times \frac{320}{60}\right)^2 \times 0.15 = 429.26\ \text{N}$$

For lowest position,

$$W + S_2 = 2F_{c2} \times \frac{a}{b}$$

$$0 + S_2 = 2 \times 301.82 \times \frac{120}{80}$$

$$S_2 = 905.46\ \text{N}$$

For highest position,

$$W + S_1 = 2F_{c1} \times \frac{a}{b}$$

$$0 + S_1 = 2 \times 429.26 \times \frac{120}{80}$$

$$S_1 = 1288.78\ \text{N}$$

(b) Stiffness of spring,

$$k = \frac{S_1 - S_2}{h}$$

$$= \frac{1288.78 - 905.46}{20} = 19.116\ \text{N/mm}$$

Gravity and spring-controlled governor

The gravity and spring-controlled governor is shown in Fig.8.5(a). This type of governor has the pivots for the bell crank levers on the moving sleeve. The spring is compressed between the sleeve and the cap which is fixed to the end of the governor shaft. As the rollers of the bell crank lever press on the cap, the sleeve is lifted against the spring compression. The forces acting on half the governor are shown in Fig.8.5(b). Taking moments about D, we have

$$F_c c = w(d + e) + (W + S)\frac{d}{2}$$

$$W + S = \frac{2[F_c c - w(d + e)]}{d} \tag{8.15}$$

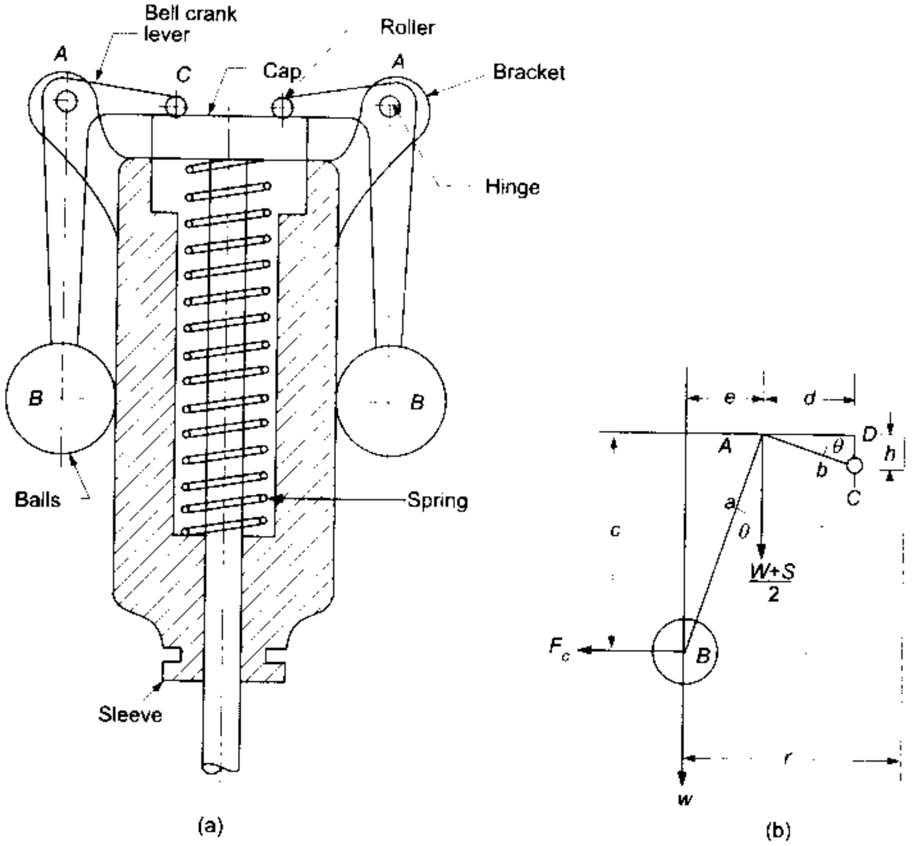


Fig.8.5 Governor with gravity and spring control

Example 8.6

In a gravity and spring controlled governor, the weight of each ball is 15 N, and the weight of the sleeve is 80 N. The two arms of the bell crank lever are at right angles and their lengths are AB = 100 mm and AC = 50 mm. The distance of the fulcrum A of each lever from the axis of rotation is 50 mm. The minimum radius of rotation of the governor balls is also 50 mm, with a corresponding equilibrium speed of 250 rpm. The sleeve lift is 15 mm for an increase in speed of 5%. Find the stiffness and initial compression of the spring.

■ **Solution**

Here w = 15 N, W = 80 N, a = 100 mm, b = 50 mm, r = 50 mm, r_2 = 50 mm, N_2 = 250 rpm, h = 15 mm, N_1 = 1.05 N_2.

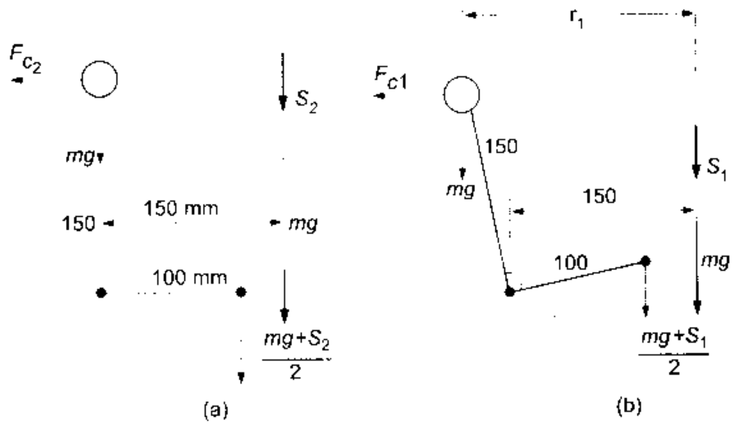


Fig.8.6 Gravity and spring-controlled governor

From Fig.8.6, we have

$$F_{c2} = \left(\frac{w}{g}\right)\omega_2^2 r_2 = \left(\frac{15}{9.81}\right)\left(2\pi \times \frac{250}{60}\right)^2 \times 0.05 = 52.4 \text{ N}$$

$$F_{c2} \times a = \left[w + \frac{W + S_2}{2}\right] b$$

$$52.4 \times 100 = \left[15 + \frac{80 + S_2}{2}\right] \times 50$$

$$5240 = 750 + 2000 + 25S_2$$

$$S_2 = 99.6 \text{ N}$$

$$r_1 = r_2 + h\frac{a}{b} = 50 + 15 \times \frac{100}{50} = 80 \text{ mm}$$

$$N_1 = 1.05 \times 250 = 262.5 \text{ rpm}$$

$$F_{c1} = \left(\frac{15}{9.81}\right)\left(\frac{2\pi \times 262.5}{60}\right)^2 \times 0.08 = 92.43 \text{ N}$$

$$W + S_1 = 2\frac{[F_{c1}c - w(d + e)]}{d}$$

$$d = b\cos\theta, e = a\sin\theta, \quad \text{and} \quad c = a\cos\theta$$

$$\sin\theta = \frac{h}{b} = \frac{15}{50} = 0.3, \quad \text{and} \quad \cos\theta = 0.954$$

$$d = 50 \times 0.954 = 48.7 \text{ mm}$$

$$e = 100 \times 0.3 = 30 \text{ mm}$$

$$c = 100 \times 0.954 = 95.4 \text{ mm}$$

$$80 + S_1 = 2\frac{[92.43 \times 95.4 - 15(48.7 + 30)]}{48.7} = 320.85$$

$$S_1 = 240.85 \text{ N}$$

Stiffness of spring, $k = \frac{S_1 - S_2}{h} = \frac{240.85 - 99.6}{15} = 9.41$ N/mm

Wilson–Hartnell governor In this governor, the balls are connected by a spring in tension, as shown in Fig.8.7(a). An auxiliary spring is attached to the sleeve mechanism through a lever by means of which the equilibrium speed for a given radius may be adjusted. The forces acting on the governor are shown in Fig.8.7(b).

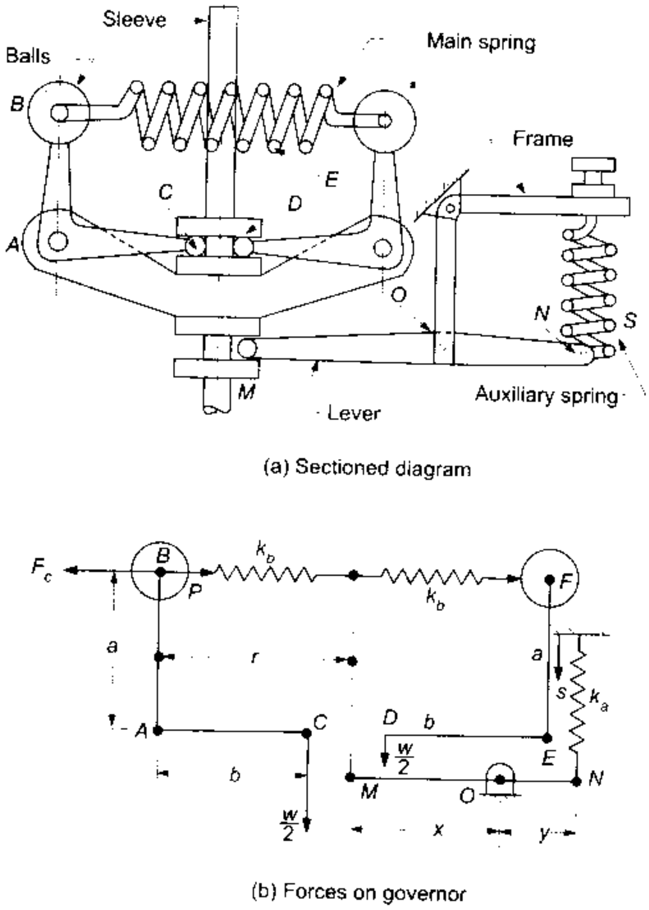


Fig.8.7 Wilson–Hartnell governor

Let w = weight of each ball

W = weight of sleeve

P = combined pull of the ball springs

S = pull of the auxiliary spring

k_a = stiffness of auxiliary spring

k_b = stiffness of each ball spring

r = radius of the balls

F_c = centrifugal force of each ball

a, b = lengths of the arms of the bell crank lever

x, y = distances of hinge O for the lever from M and N, respectively.

Total downward force on the sleeve $= W + \dfrac{Sy}{x}$

Taking moments about the fulcrum *A* of the bell crank lever, and neglecting the pull of gravity on the balls, we have

$$(F_c - P)a = \left(W + \frac{Sy}{x}\right)\frac{b}{2}$$

If suffices 1 and 2 refer to maximum and minimum equilibrium speeds respectively, then,

$$(F_{c1} - P_1)\,a = \left(W + \frac{S_1 y}{x}\right)\frac{b}{2}$$

$$(F_{c2} - P_2)\,a = \left(W + \frac{S_2 y}{x}\right)\frac{b}{2}$$

Subtracting, we get

$$[(F_{c1} - F_{c2}) + (P_2 - P_1)]\,a = \frac{(S_1 - S_2)\,yb}{2x}$$

If the radius increases from r_2 to r_1, the ball springs extend by the amount $2(r_1 - r_2)$ and the auxiliary spring extends by the amount $\dfrac{(r_1 - r_2)by}{ax}$.

$$P_1 - P_2 = 4k_b\,(r_1 - r_2)$$

and

$$S_1 - S_2 = k_a by\left[\frac{r_1 - r_2}{ax}\right]$$

$$F_{c1} - F_{c2} = 4k_b\,(r_1 - r_2) + k_a\left(\frac{by}{ax}\right)^2\frac{(r_1 - r_2)}{2}$$

or

$$4k_b + k_a\frac{\left(\frac{by}{ax}\right)^2}{2} = \frac{F_{c1} - F_{c2}}{r_1 - r_2} \tag{8.16}$$

If $k_a = 0$, then

$$k_b = \frac{F_{c1} - F_{c2}}{4(r_1 - r_2)} \tag{8.17}$$

Example 8.7

Two springs of the Wilson–Hartnell governor are designed for a tension of 1 kN in each. The weight of each ball is 75 N. In the mean position, the radius of the governor balls is 125 mm and the speed is 600 rpm. Find the tension in the auxiliary spring for this position.

When the sleeve moves up 20 mm, the speed is to be 650 rpm. Find the stiffness of the auxiliary spring, if the stiffness of each spring is 10 N/mm.

Take $a = 100$ mm, $b = 90$ mm, $x = 80$ mm and $y = 160$ mm.

■ **Solution**

At minimum speed, from Fig.8.8, we have

$$F_{c2} = \left(\frac{w}{g}\right)\omega_2^2 r_2$$

$$= \left(\frac{75}{9.81}\right)\left(2\pi \times \frac{600}{60}\right)^2 \times 0.125 = 3772.78 \text{ N}$$

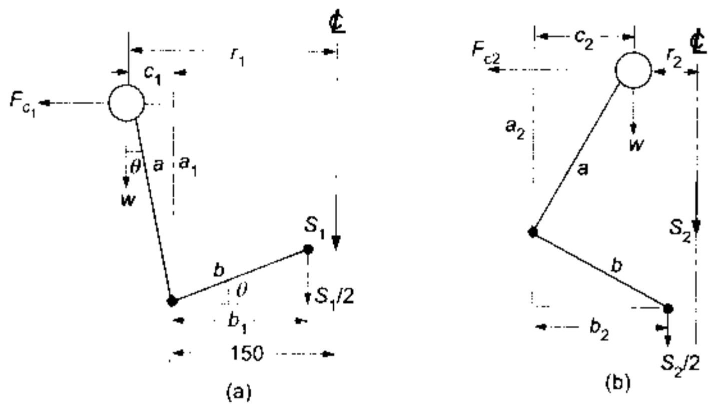


Fig.8.8 Wilson–Hartnell governor

Total pull of the ball springs, $P = 2$ kN

Taking moments about the fulcrum *A*, we have

$$F_{c2} \times a = P \times a + W \times \frac{b}{2}$$

$$3772.78 \times 100 = 2000 \times 100 + W \times \frac{90}{2}$$

or

$$W = 3939.5 \text{ N}$$

Taking moments about point *O*, we have

$$S_2 \times y = W \times x$$

$$S_2 = \frac{3939.5 \times 80}{160} = 1969.75 \text{ N}$$

Let k_a be the stiffness of auxiliary spring and

$$h = (r_1 - r_2)\frac{b}{a}$$

or

$$r_1 = r_2 + h\frac{a}{b} = 125 + \frac{20 \times 100}{90} = 147.22 \text{ mm}$$

$$F_{c1} = \left(\frac{w}{g}\right)\omega_1^2 r_1$$

$$= \left(\frac{75}{9.81}\right)\left(2\pi \times \frac{650}{60}\right)^2 \times 147.22 \times 10^{-3} = 5214.86 \text{ N}$$

Extension of the spring $= 2(r_1 - r_2) \times$ number of springs

$= 2(147.22 - 125) \times 2 = 88.88$ mm

Total spring force $= 2000 + 88.88 \times 10 = 2888.8$ N

Taking moments about *A*, neglecting the obliquity of arms, we have

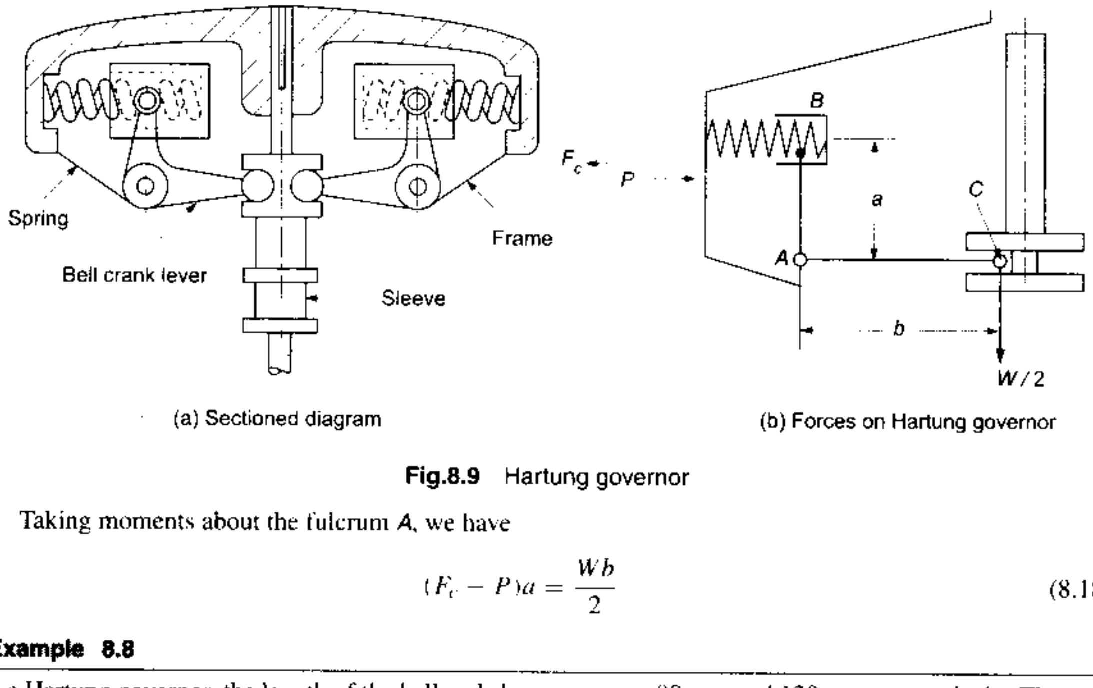
$$F_{c1} \times 100 = 2888.8 \times 100 + \left(\frac{W}{2}\right) \times 90$$

$$W = [5214.86 \times 100 - 288880]/45 = 5169 \text{ N}$$

Now taking moments about O, we have

$$S_1 \times 160 = 5169 \times 80$$
$$S_1 = 2584.5 \text{ N}$$
$$\text{Extension of auxiliary spring} = \frac{20 \times 80}{160} = 10 \text{ mm}$$
$$\text{Stiffness of auxiliary spring} = (S_1 - S_2) \,/\, \text{Extension}$$
$$= \frac{2584.5 - 1969.75}{10} = 61.47 \text{ N/mm}$$

Hartung governor The Hartung governor is shown in Fig.8.9(a). In this governor, the vertical arms of the bell crank levers are fitted with spring balls which compress against the frame of the governor when the rollers at the horizontal arm press against the sleeve. The forces acting on the governor are shown in Fig.8.9(b).



(a) Sectioned diagram

(b) Forces on Hartung governor

Fig.8.9 Hartung governor

Taking moments about the fulcrum A, we have

$$(F_c - P)a = \frac{Wb}{2} \tag{8.18}$$

Example 8.8

In a Hartung governor, the length of the ball and sleeve arms are 80 mm and 120 mm, respectively. The total travel of the sleeve is 25 mm. In the mid-position, each spring is compressed by 50 mm and the radius of rotation of the balls is 140 mm. The weight of each ball is 40 N and the spring has a stiffness of 10 N/mm. The equivalent weight of the governor gear at the sleeve is 160 N. Neglecting the moment due to the revolving masses when the arms are inclined, determine the ratio of range of speed to the mean speed of the governor. Also find the speed in the mid-position.

■ Solution

Here $a = 80$ mm, $b = 120$ mm, $h = 25$ mm, $r = 140$ mm, $w = 40$ N, $W = 160$ N, $k = 10$ N/mm, initial compression $= 50$ mm.

$$F_c = \left(\frac{w}{g}\right)\omega^2 r = \left(\frac{40}{9.81}\right)\omega^2 \times 0.14 = 0.57\omega^2 \text{ N}$$

Spring force, $S = 10 \times 50 = 500$ N

Taking moments about the fulcrum of the lever [Fig.8.10(a)], we have

$$F_c \times a = S \times a + W \times \frac{b}{2}$$

$$0.57\omega^2 \times 80 = 500 \times 80 + 160 \times \frac{120}{2}$$

$$45.6\omega^2 = 49600$$

$$\omega^2 = 1088.72$$

$$\omega = 32.98 \text{ rad/s}$$

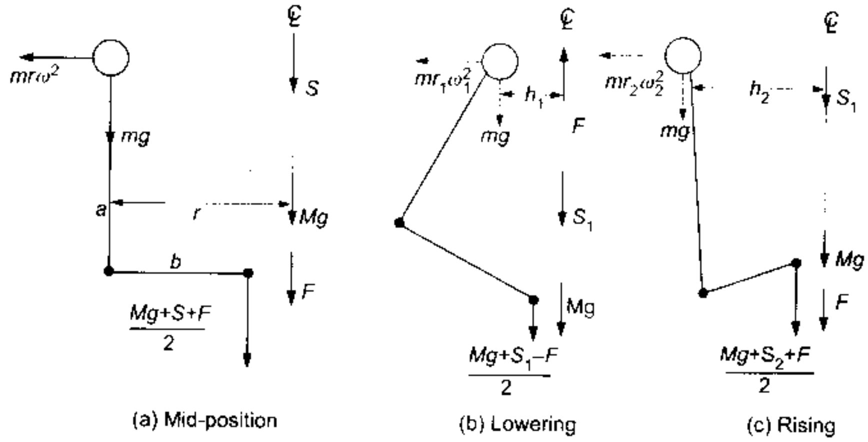


Fig.8.10 Hartung governor

Mean speed of the governor, $N = 314.9$ rpm

At maximum position [Fig.8.10(c)]

$$\frac{r_2 - r}{h} = \frac{a}{b}$$

or

$$r_2 = r + h\frac{a}{b}$$

$$= 140 + \frac{25 \times 80}{2 \times 120} = 148.3 \text{ mm}$$

$$F_{c2} = \left(\frac{w}{g}\right)\omega_2^2 r_2$$

$$= \left(\frac{40}{9.81}\right)\omega_2^2 \times 0.1483 = 0.6047\omega_1^2 \text{ N}$$

Spring force, S_2 = [Initial compression + $(r_2 - r)$] × Stiffness

$$= [50 + (148.3 - 140)] \times 10 = 583 \text{ N}$$

Taking moments about the fulcrum of the lever, we have

$$F_{c2}a = S_2 a + W\frac{b}{2}$$

$$0.6047\omega_2^2 \times 80 = 583 \times 80 + 160 \times \frac{120}{2}$$

$$48.376\omega_2^2 = 56240 = 1162.56$$

$$\omega_2 = 34.1 \text{ rad/s}$$

$$N_2 = 325.63 \text{ rpm}$$

At minimum position [Fig.8.10(b)]:

$$\frac{r - r_1}{h} = \frac{a}{b}$$

or

$$r_1 = r - h\frac{a}{b} = 140 - \frac{25 \times 80}{2 \times 120} = 131.67 \text{ mm}$$

$$F_{c1} = \left(\frac{w}{g}\right)\omega_1^2 r_1$$

$$= \left(\frac{40}{9.81}\right)\omega_1^2 \times 0.13167 = 0.537\omega_1^2 \text{ N}$$

$$S_1 = [\text{Initial compression} - (r - r_1)] \times \text{Stiffness}$$

$$= [50 - (140 - 131.67)] \times 10 = 416.7 \text{ N}$$

Taking moments about the fulcrum of the lever, we have

$$F_{c1}a = S_1 a + W\frac{b}{2}$$

$$0.537\omega_1^2 \times 80 = 416.7 \times 80 + 160 \times \frac{120}{2}$$

$$42.96\omega_1^2 = 42936$$

$$\omega_1^2 = 999.44$$

$$\omega_1 = 31.614 \text{ rad/s}$$

$$N_1 = 301.9 \text{ rpm}$$

$$\text{Range of speed} = N_2 - N_1 = 325.63 - 301.9 = 23.73 \text{ rpm}$$

$$\text{Ratio of range of speed} = \frac{\text{Range of speed}}{\text{Mean speed}} = 23.73/314.9 = 0.07536 \quad \text{or} \quad 8.536\%$$

Pickering governor The Pickering governor is shown in Fig.8.11. It consists of three straight leaf springs arranged at equal angular intervals round the spindle. Each spring carries a weight at the centre. The weights move outwards and the springs bend as they rotate about the spindle axis with increasing speed.

It is mostly used for driving gramophones.

Let m = mass attached at the centre of the leaf spring

a = distance from the spindle axis to the centre of gravity of the mass when the governor is at rest

δ = deflection of the centre of the leaf spring

ω = angular speed of the spindle

h = lift of the sleeve

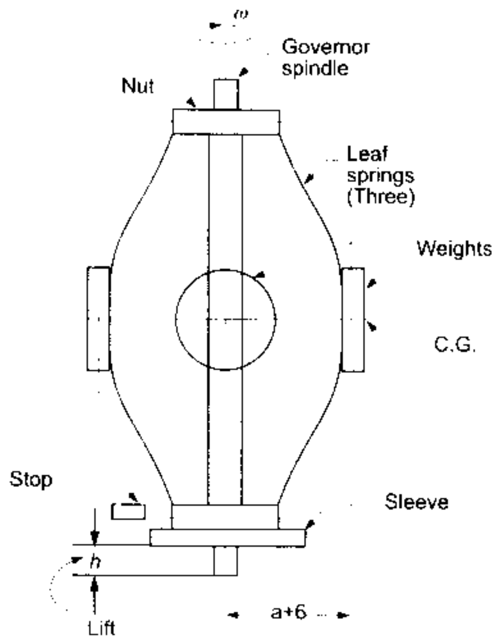


Fig.8.11 Pickering governor

The deflection of a leaf spring with both ends fixed and carrying a central load W, is given by

$$\delta = \frac{WL^3}{192EI}$$

where L = length of the spring
E = modulus of elasticity
I = moment of inertia of the spring cross-section about the neutral axis
$= \dfrac{bt^3}{12}$
b = width of the spring leaf
t = thickness of the spring leaf

In the Pickering governor, the central load is the centrifugal force.

$$W = F_c = m(a + \delta)\omega^2$$

Hence $$\delta = m(a + \delta)\omega^2 \frac{L^3}{192EI} \quad (8.19)$$

The lift of the sleeve, $$h \approx \frac{2.4\delta^2}{L} \quad (8.20)$$

Example 8.9

A gramophone is driven by a Pickering governor. The mass of each disc attached to the centre of a leaf spring is 20 g. Each spring is 5 mm wide and 0.125 mm thick. The effective length of each spring is 40 mm. The distance from the spindle axis to the centre of gravity of the mass when the governor is at rest, is 10 mm. Find the speed of the turntable when the sleeve has risen 1 mm and the ratio of the governor speed to the turntable speed is 10. Take $e = 210$ GPa.

■ Solution

Moment of inertia of the spring about the neutral axis,

$$I = \frac{bt^3}{12} = \frac{5(0.125)^3}{12} = 0.000814 \text{ mm}^4$$

Length of spring between fixed ends, $L = 40 - 1 = 39$ mm

Lift of the sleeve,

$$h = \frac{2.4\delta^2}{L}$$

$$1 = 2.4\frac{\delta^2}{39} = 0.06154\delta^2$$

$$\delta = 4.03 \text{ mm}$$

Let N = speed of the governor, and

N_t = sped of the turntable.

Then

$$\frac{N}{N_t} = 10$$

$$\delta = m\omega^2(a + \delta)\frac{L^3}{192EI}$$

$$4.03 = \frac{0.02\omega^2(10 + 4.03) \times 39^3}{192 \times 210 \times 10^3 \times 0.000814}$$

$$\omega^2 = 8.946$$

$$\omega = 2.819 \text{ rad/s}$$

$$N = 26.92 \text{ rpm}$$

$$N_t = 2.692 \text{ rpm}$$

8.3.4 Inertia Governor

The principle of the inertia governor is depicted in Fig.8.12. A mass of weight W, whose centre of gravity is at G is attached to an arm, the other end of which is pivoted at a point A on the rotating disc. The point A is selected such that points O, A and G are not collinear. The end of the arm is connected to an eccentric which operates the fuel supply valve to the prime mover.

Let v = velocity of G

r = radial distance of G from the centre of disc

Centrifugal force due to the rotating weight W, $F_c = \left(\frac{W}{g}\right) \cdot \frac{v^2}{r}$

If x = perpendicular distance of A from OG, then

Moment of F_c about $A = F_c \cdot x$

The inertia force acting on the ball perpendicular to $OG = \left(\frac{W}{g}\right) \cdot \frac{dv}{dt}$

For a governor to be rapid in action, the arm should be arranged such that as the mass moves outwards, the arm rotates in a direction opposite to that of the rotation of shaft.

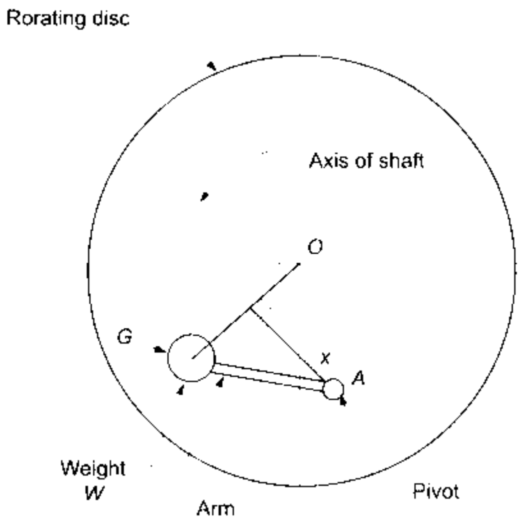


Fig.8.12 Inertia governor

8.4 PERFORMANCE OF GOVERNORS

8.4.1 Definitions

Sensitiveness For maintaining constant speed of rotation, the movement of sleeve should be as large as possible and the corresponding change of equilibrium speed as small as possible. The bigger the displacement of the sleeve for a given fractional change of speed, the more sensitive is the governor. Sensitiveness is more correctly defined as the ratio of the difference between the maximum and minimum equilibrium speeds to the mean equilibrium speed.

If N_{max} = maximum equilibrium speed

N_{min} = minimum equilibrium speed

N_{mean} = mean equilibrium speed

$$= \frac{N_{max} + N_{min}}{2}$$

Then, sensitiveness

$$= \frac{N_{max} - N_{min}}{N_{mean}}$$

$$= \frac{2\,(N_{max} - N_{min})}{N_{max} + N_{min}} \quad (8.21)$$

A more sensitive governor changes the fuel supply by a large amount when a small change in the speed of rotation takes place. This causes wide fluctuations in the engine speed, resulting in hunting of the governor.

Stability A governor is said to be stable when for each speed within the working range there is only one radius of the governor balls at which the governor is in equilibrium. For a stable governor, if the equilibrium speed increases, the radius of the governor balls must also increase.

Isochronism A governor is said to be isochronous when the equilibrium speed is constant for all radii of rotation of the balls within the working range. An isochronous governor will be infinitely sensitive. For a Porter governor,

$$\omega_1^2 = \left(\frac{g}{wh_1}\right)\left[w + \left(\frac{W}{2}\right)\cdot\left(1 + \frac{h_1}{l}\right)\right]$$

$$\omega_2^2 = \left(\frac{g}{wh_2}\right)\left[w + \left(\frac{W}{2}\right)\cdot\left(1 + \frac{h_2}{l}\right)\right]$$

For the governor to be isochronous, $N_{max} - N_{min} = 0$, or $N_{max} = N_{min}$. Therefore, h_1 equals h_2, which is not possible. Hence, a Porter governor cannot be isochronous.

For a Hartnell governor, we have

$$W + S_1 = 2F_{c1}\frac{a}{b} = 2\left(\frac{w}{g}\right)(2\pi N_{min})^2 r_1 \frac{a}{b}$$

and
$$W + S_2 = 2F_{c2}\frac{a}{b} = 2\left(\frac{w}{g}\right)(2\pi N_{max})^2 r_2 \frac{a}{b}$$

For isochronism, $N_{max} = N_{min}$. Hence

$$\frac{W + S_1}{W + S_2} = \frac{r_1}{r_2}$$

Therefore, a Hartnell governor can be isochronous.

An isochronous governor is not of much practical use, as the sleeve will move to one of its extreme positions immediately when the speed deviates from its isochronous speed.

Hunting Hunting is a condition in which the speed of the engine controlled by the governor fluctuates continuously above and below the mean speed. It is caused by a governor that is too sensitive.

Governor effort The effort of a governor is the force with which it can exert at the sleeve on the mechanism which controls the supply of fuel to the engine. The mean force exerted during the given change of speed is termed the effort. Generally effort is defined for 1% change of speed.

Power The power of a governor is defined as the work done at the sleeve for a given percentage change of speed.

$$\text{Power} = \text{Effort} \times \text{Displacement of sleeve} \tag{8.22}$$

8.4.2 Effort and Power of a Porter Governor

For the Porter governor, shown in Fig.8.2,

Let N = equilibrium speed and

c = a factor by which speed increases.

For $\alpha = \beta$, we have

$$h = \left[w + \frac{W}{w}\right]\left(\frac{g}{\omega^2}\right)$$

$$= \left[w + \frac{W}{w}\right]\left(\frac{3600\,g}{4\pi^2 N^2}\right) \tag{8.23}$$

If the speed increases to $(1 + c)N$, and the height remains the same, a downward force has to be exerted on the sleeve, then we have

$$h = \left[w + \frac{W_1}{w}\right]\left[\frac{3600g}{4\pi^2(1 + c)^2 N^2}\right] \tag{8.24}$$

where W_1 is the required sleeve load.

From (8.23) and (8.24), we have

$$W_1 + w = (W + w)(1 + c)^2$$

or
$$W_1 = (W + w)(1 + c)^2 - w$$

and
$$W_1 - W = (W + w)\left[(1 + c)^2 - 1\right]$$

Let
$$P = W_1 - W$$

= downward force which must be applied in order to prevent the sleeve from rising when increase of speed takes place.

and
$$Q = \frac{W_1 - W}{2} = \frac{P}{2}$$

= mean force exerted by the sleeve during the change of speed from N to $(1 + c)N$.

Now
$$(1 + c)^2 \approx 1 + 2c$$

Therefore,
$$P \approx 2c(W + w)$$

Governor effort,
$$Q = \frac{P}{2} \approx c(W + w) \tag{8.25}$$

If
$$x = 2(h - h_1) = \text{lift of the sleeve}$$

where
$$h_1 = \text{height corresponding to the increased speed } (1 + c)N$$
$$= \frac{h}{(1 + c)^2}$$

Then
$$x = 2h\left[1 - \frac{1}{(1 + c)^2}\right] \approx \frac{4hc}{1 + 2c}$$

Governor power
$$= Qx$$
$$\approx \left[\frac{4c^2}{1 + 2c}\right](w + W)h \tag{8.26}$$

When $\alpha \neq \beta$, then

$$Q \approx c\left[W + \frac{2w}{1 + k}\right]$$

where $k = \tan\beta/\tan\alpha$.

and
$$x \approx (1 + k)(h - h_1)$$

where
$$h_1 = \frac{h}{(1 + c)^2}$$

so that
$$x \approx (1 + k)h\left[1 - \frac{1}{(1 + c)^2}\right]$$
$$\approx (1 + k)h\left[\frac{2c}{1 + 2c}\right]$$

Governor power
$$\approx \left[\frac{2c^2}{1 + 2c}\right][W(1 + k) + 2w]h$$
$$\approx \left[\frac{4c^2}{1 + 2c}\right]\left[\frac{W(1 + k)}{2} + w\right]h \tag{8.27}$$

8.4.3 Quality of a Governor

The quality of a governor is ascertained by the following.

1. Sensitiveness, 2. Stability, 3. Effort, and 4. Power.

8.4.4 Controlling Force

When the speed of rotation is uniform, each ball of the governor is subjected either directly or indirectly to an inward pull, which is equal and opposite to the outward centrifugal reaction. This inward pull is termed the controlling force. A curve drawn to show how the pull varies with the radius of rotation of the ball is called a controlling force curve, as shown in Fig.8.13.

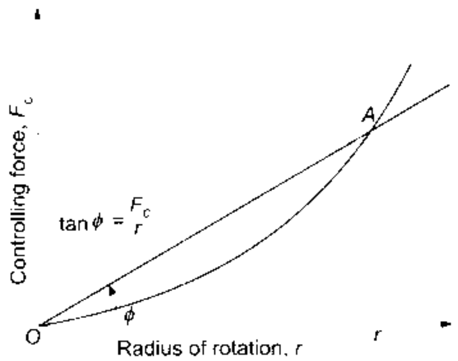


Fig.8.13 Controlling force diagram

Controlling force diagram for a Porter governor

Controlling force,
$$F_c = \left(\frac{w}{g}\right)\omega^2 r$$

or
$$\omega = \left[\frac{gF_c}{wr}\right]^{0.5}$$

$$= \left[\left(\frac{g}{w}\right)\cdot\tan\phi\right]^{0.5} \quad (8.28)$$

If controlling force curve is a straight line, then

$$\tan\phi = \left(\frac{w}{g}\right)\omega^2$$

$$= kN^2 \quad (8.29)$$

where $k = \left(\frac{w}{g}\right)\left(\frac{2\pi}{60}\right)^2 = a$ constant.

Using (8.29), the angle ϕ may be determined for different values of N, and lines are drawn from the origin, as shown in Fig.8.14. These lines enable the equilibrium speed, corresponding to a given radius of rotation, to be determined.

Controlling force diagram for spring-controlled governors

The controlling force diagram for spring-controlled governors is a straight line, as shown in Fig.8.15.

Controlling force, $F_c = \left(\frac{w}{g}\right)\omega^2 r$ or $\frac{F_c}{r} = \left(\frac{w}{g}\right)\omega^2$

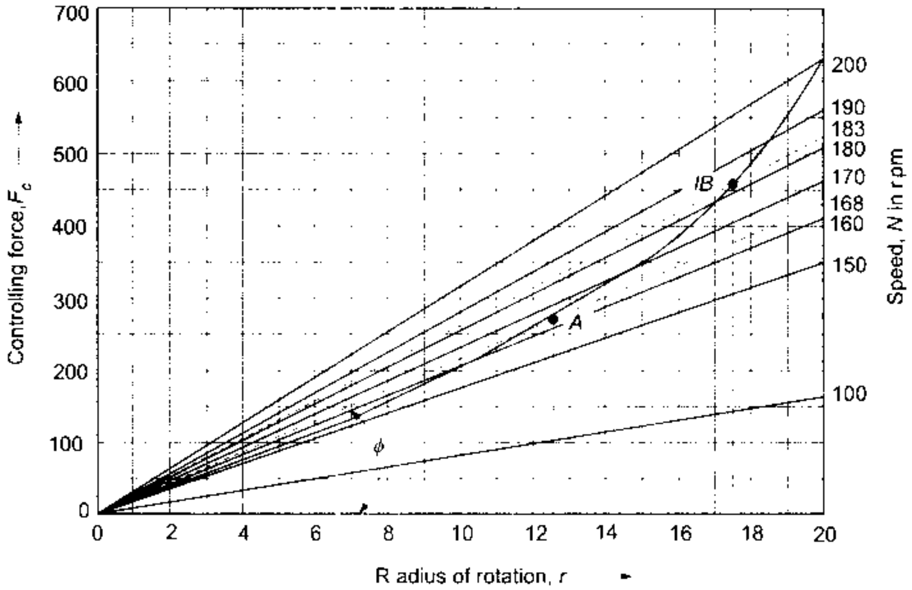


Fig.8.14 Controlling force vs Radius of rotation

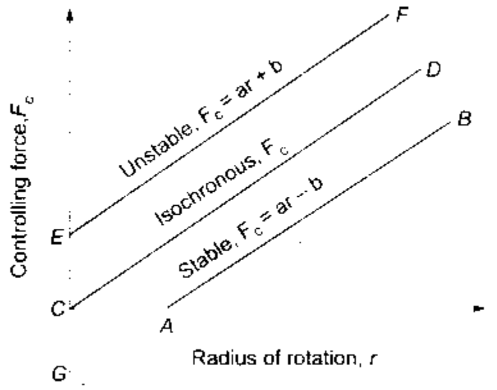


Fig.8.15 Controlling force for a spring loaded governor

The stability of a spring-controlled governor can be ascertained as follows:

1. For a stable governor, the controlling force must increase as the radius of rotation increases, that is, F_c/r must increase as r increases. Therefore, the controlling force line *AB* when produced must intersect the controlling force axis below the origin at *G*, as shown in Fig.8.15. The equation of line *AB* is given by,

$$F_c = ar - b \quad (8.30)$$

where a and b are constants.

2. When $b = 0$, the controlling force line CD passes through the origin, and the governor becomes isochronous, because F_c/r will remain constant for all radii of rotation. The equation of line CD is,

$$F_c = ar \tag{8.31}$$

3. If b is positive, then F_c/r decreases as r increases, so that the equilibrium speed of the governor decreases with an increase of the radius of rotation of the balls, which is not possible. Such a governor is said to be unstable. The equation of line EF is,

$$F_c = ar + b \tag{8.32}$$

Coefficient of insensitiveness There is always friction in the joints and operating mechanism of the governor. The frictional force opposes the motion of the sleeve. To account for the frictional forces.

Let F_s = force required at the sleeve to overcome friction
F_b = corresponding radial force required at the balls
F_c = controlling force on each ball
W = total load on the sleeve

For decrease in speed, sleeve load, $W_1 = W - F_s$
For increase in speed, sleeve load, $W_2 = W + F_s$
For decrease in speed, controlling force, $F_{c1} = F_c - F_b$
For increase in speed, controlling force, $F_{c2} = F_c + F_b$

Now
$$F_c = \left(\frac{w}{g}\right)\omega^2 r = KN^2$$
$$F_{c1} = KN_1^2$$
$$F_{c2} = KN_2^2$$

Similarly,
$$F_c + F_b = KN_1^2$$
$$F_c - F_b = KN_2^2$$

Subtracting, we get
$$\frac{2F_b}{F_c} = \frac{N_1^2 - N_2^2}{N^2} = \frac{(N_1 + N_2)(N_1 - N_2)}{N^2}$$

Now
$$N_1 + N_2 \approx 2N$$
$$\frac{F_b}{F_c} \approx \frac{N_1 - N_2}{N} \tag{8.33}$$

The coefficient of insensitiveness = F_b/F_c and is defined as the ratio of the difference of speed at ascent and descent for same radius of rotation to the steady speed at the same radius of rotation.

For a Porter governor, as shown in Fig.8.2, we have

$$F_c = \tan\alpha \left[w + \left(\frac{W}{2}\right)(1 + k)\right]$$

and
$$F_c \pm F_b = \tan\alpha \left[w + \left(\frac{W \pm F_s}{2}\right)(1 + k)\right]$$
$$F_b = \tan\alpha \cdot \frac{F_s(1 + k)}{2} \tag{8.34}$$

Similarly, for a spring loaded governor of Hartnell type (Fig.8.4), neglecting the obliquity of the arms, we have

$$F_c a = \frac{Pb}{2}$$

and

$$(F_c \pm F_b)a = (P \pm F_s)\frac{b}{2}$$

or

$$F_b = \frac{F_s b}{2a} \qquad (8.35)$$

Fig.8.16 shows the effect of friction on the controlling force diagram. We see that for one value of the radius of rotation, there are three values of controlling force, as discussed below:

1. For decreasing speed, the controlling force reduces to F_{c2} and the corresponding speed is N_2.
2. For increasing speed, the controlling force increases to F_{c1} and the corresponding speed is N_1.
3. For friction neglected, the controlling force is F_c and the corresponding speed is N.

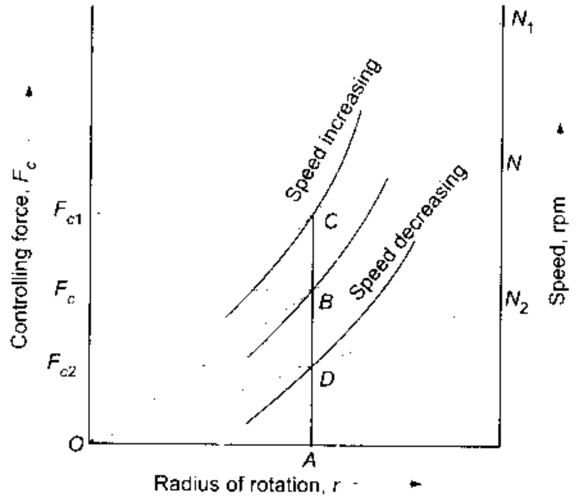


Fig.8.16 Effect of friction on the controlling force diagram

Example 8.10

A Porter governor has equal arms of 240 mm length each and pivoted on the axis of rotation. Each ball is of 50 N weight and the weight of the central load is 250 N. The radius of rotation of the ball is 150 mm when the governor begins to rise and 200 mm when the governor is at maximum speed. Find the range of speed, sleeve lift, governor effort and power of the governor, when the friction at the sleeve is neglected.

■ **Solution**

Let N_1 = maximum speed

N_2 = minimum speed

$$h_1 = \left[250^2 - 200^2\right]^{0.5} = 150 \text{ mm}$$

$$h_2 = \left[250^2 - 150^2\right]^{0.5} = 200 \text{ mm}$$

$$\omega_1^2 = \left(1 + \frac{W}{w}\right)\left(\frac{g}{h_1}\right)$$

$$= \left(1 + \frac{250}{50}\right)\left(\frac{9.81}{0.15}\right) = 392.4$$

$$\omega_1 = 19.81 \text{ rad/s}$$

$$N_1 = 189 \text{ rpm}$$

and

$$\omega_2^2 = \left(1 + \frac{W}{w}\right)\left(\frac{g}{h_2}\right)$$

$$= \left(1 + \frac{250}{50}\right)\left(\frac{9.81}{0.2}\right) = 294.3$$

$$\omega_2 = 18.15 \text{ rad/s}$$

$$N_2 = 163.8 \text{ rpm}$$

$$\text{Range of speed} = N_1 - N_2$$

$$= 189 - 163.8 = 25.2 \text{ rpm}$$

Sleeve lift,

$$x = 2\,(h_2 - h_1)$$

$$= 2(200 - 150) = 100 \text{ mm}$$

Let

$$c = \text{percentage increase in speed}$$

$$= \frac{N_1 - N_2}{N_2} = \frac{25}{163.8} = 0.1526 \text{ or } 15.26\%$$

Governor effort,

$$P = c(w + W)$$

$$= 0.1526(50 + 250) = 45.78 \text{ N}$$

$$\text{Power of the governor} = P \cdot x$$

$$= 45.78 \times 0.1 = 4.578 \text{ Nm}$$

Example 8.11

The radius of rotation of the balls of a Hartnell governor is 100 mm at the minimum speed of 300 rpm. Neglecting gravity effects, determine the speed after the sleeve has lifted by 50 mm. Also determine the initial compression of the spring, governor effort and the power. Take length of ball arm of lever = 150 mm, length of sleeve arm = 100 mm, weight of each ball = 40 N and stiffness of spring = 25 N/mm.

■ **Solution**

For maximum speed

$$h = (r_1 - r_2)\,\frac{b}{a}$$

$$r_1 = r_2 + \frac{ha}{b} = 100 + \frac{50 \times 150}{100} = 175 \text{ mm}$$

$$F_{c1} = \left(\frac{w}{g}\right)\omega_1^2 r_1 = \left(\frac{40}{9.81}\right)\left(2\pi \times \frac{N_1}{60}\right)^2 \times 0.175 = 0.007825 N_1^2 \text{ N}$$

$$F_{c2} = \left(\frac{w}{g}\right)\omega_2^2 r_2 = \left(\frac{40}{9.81}\right)\left(2\pi \times \frac{300}{60}\right)^2 \times 0.1 = 402.43 \text{ N}$$

Taking moments about the fulcrum of the lever, we have

$$F_{c2}a = (W + S_2) \times \frac{b}{2}$$

$$402.43 \times 150 = (0 + S_2) \times \frac{100}{2}$$

$$S_2 = 1208.3 \text{ N}$$

$$S_1 - S_2 = hk$$

$$S_1 = 1208.3 + 50 \times 25 = 2458.3 \text{ N}$$

$$F_{c1}a = (W + S_1) \times \frac{b}{2}$$

$$0.007825 N_1^2 \times 150 = (0 + 2458.3) \times \frac{100}{2}$$

$$N_1 = 323.54 \text{ rpm}$$

$$\text{Initial compression of spring} = \frac{S_2}{k} = \frac{1208.3}{25} = 48.29 \text{ mm}$$

Governor effort,
$$P = \frac{S_1 - S_2}{2} = \frac{2458.3 - 1208.3}{2} = 625 \text{ N}$$

$$\text{Governor power} = Ph = 625 \times 0.05 = 31.25 \text{ Nm}$$

Example 8.12

A Porter governor has equal arms 250 mm long pivoted on the axis of rotation. The weight of each ball is 25 N and the weight of the sleeve is 150 N. The ball path is 120 mm when the governor begins to lift and 150 mm at the maximum speed. Determine the range of speed. If the friction at the sleeve is equivalent to a force of 15 N, find the coefficient of insensitiveness.

Solution

At maximum speed,
$$h_1 = \left[250^2 - 150^2\right]^{0.5} = 200 \text{ mm}$$

At minimum speed,
$$h_2 = \left[250^2 - 120^2\right]^{0.5} = 219.32 \text{ mm}$$

$$\omega_1^2 = \left(1 + \frac{W}{w}\right)\left(\frac{g}{h_1}\right) = \left(1 + \frac{150}{25}\right)\left(\frac{9.81}{0.2}\right) = 343.35$$

$$\omega_1 = 18.53 \text{ rad/s}$$

$$N_1 = 176.95 \text{ rpm}$$

$$\omega_2^2 = \left(1 + \frac{W}{w}\right)\left(\frac{g}{h_2}\right) = \left(1 + \frac{150}{25}\right)\left(\frac{9.81}{0.21932}\right) = 313.1$$

$$\omega_2 = 18.69 \text{ rad/s}$$

$$N_2 = 168.97 \text{ rpm}$$

$$\text{Range of speed} = N_1 - N_2 = 176.95 - 168.97 = 8.98 \text{ rpm}$$

$$\text{Coefficient of insensitiveness} = \frac{F}{w + W} = \frac{15}{25 + 150} = 0.0857 \text{ or } 8.57\%$$

Example 8.13

In a spring-controlled governor, the curve of controlling force is a straight line. When balls are 400 mm apart, the controlling force is 1200 N, and when 200 mm apart, 450 N. At what speed will the governor run when the balls are 250 mm apart. What initial tension on the spring would be required for isochronism and what would then be the speed? The weight of each ball is 100 N.

■ **Solution**

$$F_{c1} = 1200 \text{ N}, \quad r_1 = 400 \text{ mm}$$
$$F_{c2} = 450 \text{ N}, \quad r_2 = 200 \text{ mm}$$
$$w = 100 \text{ N}$$

For the stability of the spring-controlled governor, we have

$$F_c = ar - b$$
$$1200 = 0.2a - b$$
$$450 = 0.1a - b$$

Solving for a and b, we get

$$a = 7500, \quad b = 300$$
$$F_c = 7500\,r - 300$$

For $r = 125$ mm, $F_c = 7500 \times 0.125 - 300 = 638.5$ N

$$F_c = \left(\frac{w}{g}\right)\left(\frac{2\pi N}{60}\right)^2 r$$
$$638.5 = \left(\frac{100}{9.81}\right)\left(\frac{2\pi \times N}{60}\right)^2 \times 0.125$$
$$N = 213.6 \text{ rpm}$$

For an isochronous governor, $b = 0$. Therefore, the initial tension required is 300 N.

$$F_c = ar$$
$$\left(\frac{w}{g}\right)\omega^2 r = ar$$

or

$$\omega^2 = a\frac{g}{w} = 7500 \times \frac{9.81}{100} = 735.75$$
$$\omega = 28.125 \text{ rad/s}$$
$$N = 259 \text{ rpm}$$

Exercises

1 A porter governor has two balls of 25 N weight each and a central load of 150 N. The arms are 200 mm long, pivoted on the axis. If the maximum and minimum radii of rotation of the balls are 150 mm and 120 mm respectively, find the range of speed.

2 A loaded governor of the Porter type has equal arms and links each 300 mm long. The weight of each ball is 20 N and the central weight is 120 N. When the ball radius is 150 mm, the valve is fully open and when the radius is 180 mm, the valve is closed. Find the maximum speed and the range of speed. If the maximum speed is to be increased 25% by an addition of weight to the central load, find its value.

3 The arms of a Porter governor are 250 mm long. The upper arms are pivoted on the axis of rotation and the lower arms are attached to the sleeve at a distance of 40 mm from the axis of rotation. The load on the sleeve is 525 N and the weight of each ball is 75 N. Determine the equilibrium speed when the radius of the balls is 200 mm. What will be the range of speed for this position, if the frictional resistance to the motion of the sleeve is equivalent to a force of 30 N?

4 A Proell governor has all the four arms of length 300 mm. The upper and lower ends of the arms are pivoted on the axis of rotation of the governor. The extension arms of the lower links are each 100 mm long and parallel to the axis when the radius of the ball path is 150 mm. The weight of each ball is 45 N and the weight of the central load is 350 N. Determine the equilibrium speed of the governor.

5 A spring-controlled governor of the Hartnell type with a central spring under compression has balls of weight 20 N. The ball and sleeve arms of the bell crank lever are respectively 120 mm and 80 mm long and are right angles. In the lower position of the sleeve, the radius of rotation of the balls is 80 mm and the ball arms are parallel to the governor axis. Find the initial load on the spring in order that the sleeve may begin to lift at 325 rpm. If the stiffness of the spring is 25 N/mm, what is the equilibrium speed corresponding to a sleeve lift of 15 mm?

6 A governor of the Hartnell type has each ball of weight 15 N and the lengths of vertical and horizontal arms of the bell crank lever are 120 mm and 60 mm respectively. The fulcrum of the bell crank lever is at a distance of 100 mm from the axis of rotation. The maximum and minimum radii of rotation of the balls are 120 mm and 80 mm and the corresponding equilibrium speeds are 325 rpm and 300 rpm, respectively. Find the stiffness of the spring and the equilibrium speed when the radius of rotation is 100 mm.

7 A Hartnell governor balls are of 30 N weight each. The balls radius is 120 mm in the mean position when the ball arms are vertical and the speed is 150 rpm, with the sleeve rising. The length of the ball arms is 150 mm and the sleeve arm 100 mm. The stiffness of the spring is 8 N/mm and the total sleeve movement is 15 mm from the mean position. Allowing for a constant frictional force of 15 N acting at the sleeve, determine the speed range of the governor in the lowest and highest sleeve positions. Neglect the obliquity of the ball arms.

8 In a spring-controlled governor of the Hartung type, the lengths of the horizontal and vertical arms of the bell crank lever are 120 mm and 90 mm respectively. The fulcrum of the bell crank lever is at a distance of 120 mm from the axis of rotation of the governor. Each revolving weight is 100 N. The stiffness of the spring is 25 N/mm. If the length of each spring is 120 mm when the radius of rotation is 75 mm and the equilibrium speed is 350 rpm, find the free length of the spring. If the radius of rotation increases to 120 mm. What will be the corresponding percentage increase in speed?

9 The following particulars refer to a Wilson–Hartnell governor:

Weight of each ball = 50 N

Minimum radius = 100 mm

Maximum radius = 120 mm

Minimum speed = 240 rpm

Maximum speed = 256 rpm

Length of ball arm of lever = 80 mm

Length of sleeve arm of lever = 60 mm

Combined stiffness of ball springs = 0.75 N/mm

Find the stiffness of the auxiliary spring, if the lever is pivoted at the middle position.

10 A Porter governor has arms 250 mm long. The upper arms are pivoted on the axis of rotation and the lower arms are attached to a sleeve at a distance of 30 mm from the axis. Each ball has a weight of 20 N and the weight of the load on the sleeve is 200 N. If the radius of rotation of the balls at a speed of 260 rpm is 100 mm, find the speed of the governor after the sleeve has lifted 50 mm. Also determine the effort and power of the governor.

11 The upper arms of a Porter governor are pivoted on the axis of rotation and the lower arms are pivoted to the sleeve at a distance of 40 mm from the axis of rotation. The length of each arm is 350 mm and the weight of each ball is 50 N. If the equilibrium speed is 240 rpm, when the radius of rotation is 200 mm, find the weight of the sleeve. If the friction is equivalent to a force of 50 N at the sleeve, find the coefficient of insensitiveness at 200 mm radius.

12 A Pickering governor employed in a gramophone consists of three leaf springs each 40 mm long, 5 mm wide and 1.5 mm thick. Each of the springs has disc of mass 20 g, attached to the centre. The distance between the axis of the governor spindle and the centre of gravity of the disc when at rest is 10 mm. Find the equilibrium speed of the turntable to which this governor is fixed, if the ratio of the governor speed to the speed of the turntable is 10 and the lift of the sleeve is 0.75 mm. Take modulus of elasticity of leaves as 210 GPa.

13 In an inertia governor the disc has two weights, each weighing 50 N, attached to the ends of a 300 mm long rod. The rod is pivoted at its centre to the disc such that the pivot falls on the vertical axis of the disc. The rod is horizontal in its neutral position and is 50 mm from the horizontal axis of the disc. Each of the weights is circular in shape of 50 mm radius. Find the torque required about the pivot of the rod if the disc revolves at 300 rpm and receives an angular acceleration of 1 rad/s^2.

14 The controlling force Fc for a spring loaded governor is given by $F_c = 300\ r - 80$ N where r is in mm. The weight of each ball is 50 N and the extreme radii of rotation of the balls are 100 mm and 175 mm respectively. Find the maximum and minimum equilibrium speeds. If the friction of the governor mechanism is equivalent to a radial force of 5 N at each ball, find the extent to which the equilibrium speeds are affected at the extreme radii of rotation.

15 (a) Explain the difference between hunting and stability of a governor. (b) Discuss the requirements for satisfactory performance of a centrifugal governor. (c) Define controlling force, stability, sensitiveness and hunting with reference to governors.

16 (a) State the different types of governors. What is the difference between centrifugal and inertia type governors. Why is the former preferred to the latter?

(b) Prove that the sensitiveness of a Proell governor is greater than that of a Porter governor.

(c) A Proell governor has arms of 305 mm length. The upper arms are hinged on the axis of rotation, whereas the lower arms are pivoted at a distance of 38 mm from the axis of rotation. The extension of lower arms to which the balls are attached are 102 mm long. Each ball mass is 4.8 kg and the load on the sleeve is 54 kg. At minimum radius of rotation of 165 mm, the extensions are parallel to the governor axis, determine the equilibrium speed at radii 165 mm and 216 mm.

17 Explain the phenomenon of hunting in centrifugal governors. Calculate the natural period of oscillation of the governor balls considering Hartnell governor.

18 (a) Explain the use of controlling force curves in the determination of stability of a centrifugal governor.

(b) Define effort and power in relation to governors. Obtain the expressions for these in the case of a Porter governor.

(c) Deduce the condition under which a centrifugal governor becomes unstable.

19 A centrifugal governor shown in Fig.8.17 has two masses each of weight w connected by a helical spring. The arms carrying the weights are parallel to the axis of rotation at the speed of 900 rpm. If the speed is increased by 1%, it requires a force of 30 N to maintain the sleeve at the same position. Determine (a) the value of the masses, (b) the stiffness of the spring and its initial extension if the sleeve moves by 10 mm for a change of speed of 250 rpm.

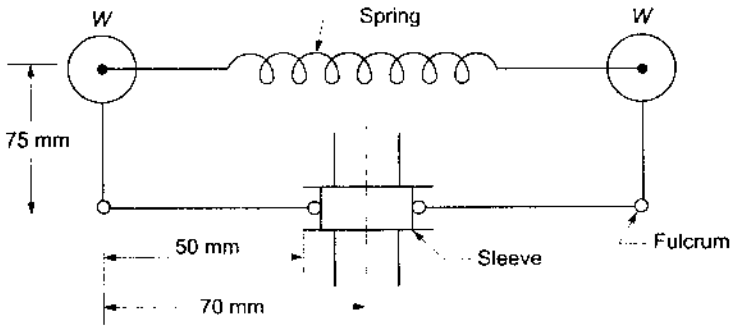


Fig.8.17

20 A loaded Porter governor has four links each 250 mm long, two revolving masses each weighing 30 N and a central dead weight weighing 200 N. All the links are attached to respective sleeves at radial distances of 40 mm from the axis of rotation. The masses revolve at a radius of 150 mm at minimum speed and at a radius of 200 mm at maximum speed. Determine the range of speed.

21 A Watt governor has an arm of uniform section of length L and mass m and a ball of mass M. Show that when revolving with angular velocity ω, it makes an angle θ to the vertical, where

$$\cos\theta = g(M + m/2)/\omega^2 L(M + m/3)$$

Also determine the angle θ for the case when the bar is not of uniform cross-section, its radius of gyration about the point of attachment being k and distance of centre of gravity to the point of attachment being d.

22 In the loaded Proell governor shown in Fig.8.18 each ball weighs 3 kg and the central sleeve weighs 25 kg. The arms are of 200 mm length and pivoted about axes displaced from the central axis of rotation by 38.5 mm, y = 238 mm, x = 303.5 mm, CE = 85 mm, MD = 142.5 mm. Determine the equilibrium speed.

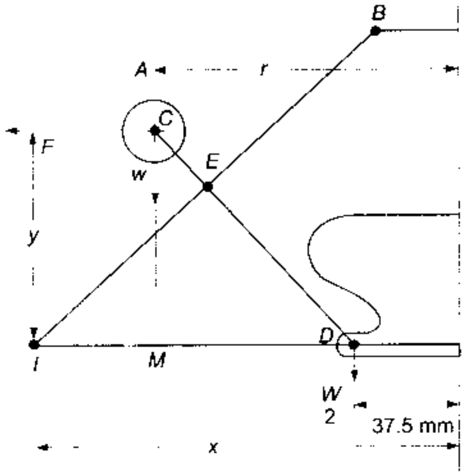


Fig.8.18

23 The arms of a Porter governor are pivoted on the governor axis and are each 250 mm long. The mass of each ball is 0.5 kg and the mass of the sleeve is 2 kg. The arms are inclined at an angle of 30° to the governor axis in the lowermost position of the sleeve. The lift is equal to 50 mm. Determine the force of friction if the speed at the moment the sleeve starts lifting from the lowermost position is the same as the speed at the moment it falls from the uppermost position.

24 In a spring-controlled governor of the Hartung type, the length of the horizontal and vertical arms of the bell crank levers are 100 mm and 80 mm respectively. The fulcrum of the bell crank lever is at a distance of 120 mm from the axis of the governor. Each revolving mass is 8 kg. The stiffness of the spring is 20 kN/m. If the length of each spring is 120 mm, when the radius of rotation is 70 mm and the equilibrium speed is 380 rpm, find the free length of the spring. If the radius of rotation increases to 120 mm, what will be the corresponding percentage increase in speed? Ignore sleeve mass.

25 The length of the arms of a Porter governor are 300 mm long. The upper and lower arms are pivoted to links at 50 mm and 60 mm respectively from the axis of rotation. The mass of each ball is 5 kg and the sleeve is of mass 60 kg. The frictional force on the sleeve is 35 N. Determine the range of speed for extreme radii of rotation of 120 mm and 150 mm.

26 In a Porter governor, all four arms are of equal length of 250 mm and are hinged on the spindle axis. The mass of each ball is 2.5 kg and the sleeve mass is 25 kg. The force of friction at the sleeve is 30 N. The inclination of arms to spindle axis is 30° and 45° in the lowest and highest position respectively. Calculate (a) the sleeve lift, (b) speeds at the bottom, middle and top of the sleeve by neglecting and considering friction.

27 The following data refers to a Proell governor:

Mass of each ball = 5 kg

Mass of sleeve = 60 kg

Length of each arm = 250 mm

Distance of pivots of lower arms from axis of rotation = 30 mm

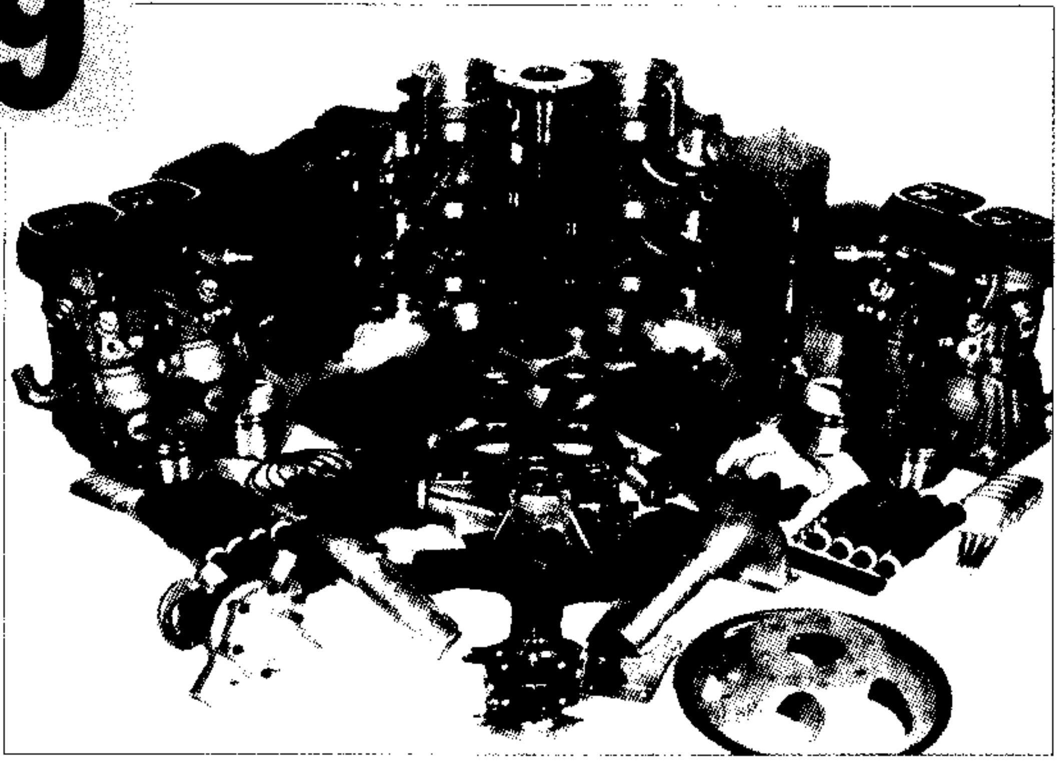
Length of extensions of lower arms = 100 mm

The extension arms are parallel to the axis of the governor at the minimum radius. Determine the equilibrium speeds corresponding to extreme radii of 160 mm and 220 mm.

28 In a spring-controlled governor, the controlling force curve is a straight line. The balls are 450 mm apart when the controlling force is 1450 N and 250 mm when it is 750 N. The mass of each ball is 8 kg. Determine the speed at which the governor runs when the balls are 300 mm apart. By how much should the initial tension be increased to make the governor isochronous? Also find the isochronous speed.

29 In a Porter governor, each arm is 250 mm long and is pivoted at the axis of rotation. The mass of each ball is 4.5 kg and the load on the sleeve is 25 kg. The extreme radii of rotation are 100 mm and 150 mm. Plot a graph of controlling force versus the radius of rotation and set off a speed scale along the ordinate corresponding to a radius of 150 mm.

9



INERTIA FORCE AND TURNING MOMENT

9.1 INTRODUCTION

Inertia force arises due to the mass of the reciprocating parts, whereas turning moment arises due to crank effort. In this chapter, we shall study about the inertia force in mechanisms, reciprocating parts, turning moment diagrams and flywheels.

9.2 MOTION ANALYSIS OF RECIPROCATING MECHANISM

Consider the slider crank mechanism, as shown in Fig.9.1. *OC* is the crank, *BC* the connecting rod, *C* the crank pin and *B* the gudgeon pin or the cross head. The crank is rotating clockwise with angular speed ω.

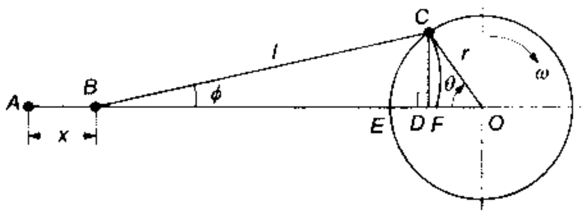


Fig.9.1 Slider–crank mechanism

9.2.1 Velocity and Acceleration of Piston

Displacement of the piston from top dead centre,

$$\begin{aligned} x &= AB = EF = ED + DF \\ &= (OE - OD) + (BF - BD) \\ &= (r - r\cos\theta) + (l - l\cos\phi) \\ &= r(1-\cos\theta) + l(1-\cos\phi) \end{aligned}$$

Now $\quad CD = r\sin\theta = l\sin\phi$

or $\quad \sin\phi = \left(\frac{r}{l}\right)\sin\theta = \frac{\sin\theta}{n}$

where $l/r = n$ is the ratio of the length of the connecting rod to that of the crank.

$$\cos\phi = [1 - \sin^2\phi]^{0.5} = \frac{(n^2 - \sin^2\theta)^{0.5}}{n}$$

Therefore,

$$\begin{aligned} x &= r(1-\cos\theta) + l\left[\frac{1 - (n^2 - \sin^2\theta)^{0.5}}{n}\right] \\ &= r\left[(1-\cos\theta) + \left\{n - \left(n^2 - \sin^2\theta\right)^{0.5}\right\}\right] \end{aligned} \tag{9.1}$$

Velocity of the piston,

$$\begin{aligned} v_p &= \frac{dx}{dt} = \left(\frac{dx}{d\theta}\right)\cdot\left(\frac{d\theta}{dt}\right) = \omega\cdot\frac{dx}{d\theta} \\ &= \omega r\left[\sin\theta + \left(\frac{1}{2}\right)\left(n^2 - \sin^2\theta\right)^{-0.5}\cdot 2\sin\theta\cos\theta\right] \\ &= \omega r\left[\frac{\sin\theta + \sin 2\theta}{\left\{2\left(n^2 - \sin^2\theta\right)^{0.5}\right\}}\right] \\ &\approx \omega r\left[\frac{\sin\theta + \sin 2\theta}{2n}\right] \end{aligned} \tag{9.2}$$

Acceleration of the piston, $\quad a_p = \frac{dv}{dt} = \left(\frac{dv}{d\theta}\right)\cdot\left(\frac{d\theta}{dt}\right) = \omega\cdot\frac{dv}{d\theta}$

$$= \omega^2 r \left[\frac{\cos\theta + 0.5\left\{\sin 2\theta \times 05\left(n^2 - \sin^2\theta\right)^{-0.5}(-2\sin\theta\cos\theta) - \left(n^2 - \sin^2\theta\right)^{0.5} \cdot 2\cos 2\theta\right\}}{\left(n^2 - \sin^2\theta\right)} \right]$$

$$= \omega^2 r \left[\frac{\cos\theta - \left\{0.25\sin^2 2\theta + \cos 2\theta\left(n^2 - \sin^2\theta\right)\right\}}{\left(n^2 - \sin^2\theta\right)^{1.5}} \right]$$

$$\approx \omega^2 r \left[\frac{\cos\theta + \cos 2\theta}{n} \right] \qquad (9.3)$$

9.2.2 Angular Velocity and Acceleration of Connecting Rod

Now

$$\sin\phi = \frac{\sin\theta}{n}$$

$$\cos\phi \cdot \frac{d\phi}{dt} = \frac{\cos\theta}{n \cdot d\theta/dt}$$

Angular velocity of connecting rod,

$$\omega_r = \frac{d\phi}{dt} = \left(\frac{\cos\theta}{\cos\phi}\right) \cdot \left(\frac{\omega}{n}\right)$$

$$= \frac{\omega\cos\theta}{\left(n^2 - \sin^2\theta\right)^{0.5}} \qquad (9.4)$$

Angular acceleration of connecting rod,

$$\alpha_r = \frac{d^2\phi}{dt^2} = \left(\frac{d}{d\theta}\right)\left(\frac{d\phi}{dt}\right) \cdot \left(\frac{d\theta}{dt}\right) = \omega \cdot \left(\frac{d}{d\theta}\right)\left(\frac{d\phi}{dt}\right)$$

$$= \omega^2 \left[\frac{-\cos\theta \times 0.5\left(n^2 - \sin^2\theta\right)^{-0.5}(-2\sin\theta\cos\theta) - \left(n^2 - \sin^2\theta\right)\sin\theta}{\left(n^2 - \sin^2\theta\right)^{1.5}} \right]$$

$$= -\omega^2 \left[\frac{-\sin\theta\cos^2\theta + \sin\theta\left(n^2 - \sin^2\theta\right)}{\left(n^2 - \sin^2\theta\right)^{1.5}} \right]$$

$$= -\omega^2 \sin\theta \left[\frac{-\cos^2\theta + n^2 - \sin^2\theta}{\left(n^2 - \sin^2\theta\right)^{1.5}} \right]$$

$$= -\omega^2 \sin\theta \left[\frac{n^2 - 1}{\left(n^2 - \sin^2\theta\right)^{1.5}} \right] \qquad (9.5a)$$

$$\approx \frac{-\omega^2 \sin\theta}{n} \qquad (9.5b)$$

Example 9.1

The length of connecting rod of a gas engine running at 340 rpm is 600 mm and the crank is 120 mm long. When the piston has moved 1/4th stroke during outstroke, determine (a) the angular position of crank, (b) the angular speed of connecting rod and (c) the acceleration of the piston.

Solution

(a)

$$n = \frac{l}{r} = \frac{600}{120} = 5$$

$$\omega = 2\pi \times \frac{340}{60} = 35.6 \text{ rad/s}$$

$$r = \frac{L}{2}, \quad \text{where } L \text{ is the stroke length}$$

$$x = r\left[(1-\cos\theta) + \left\{n - \left(n^2 - \sin^2\theta\right)^{0.5}\right\}\right]$$

$$\frac{x}{L} = \frac{x}{2r} = 0.5\left[(1-\cos\theta) + \left\{n - \left(n^2 - \sin^2\theta\right)^{0.5}\right\}\right]$$

$$\frac{1}{4} = 0.5\left[(1-\cos\theta) + \left\{n - \left(n^2 - \sin^2\theta\right)^{0.5}\right\}\right]$$

$$0.5 = 1 - \cos\theta + 5 - \left(25 - \sin^2\theta\right)^{0.5}$$

$$5.5 = \cos\theta + \left(25 - \sin^2\theta\right)^{0.5}$$

or

$$(5.5 - \cos\theta)^2 = 25 - \sin^2\theta$$

$$30.25 + \cos^2\theta - 11\cos\theta = 25 - \sin^2\theta$$

$$6.25 - 11\cos\theta = 0$$

$$\cos\theta = 0.5682$$

$$\theta = 55.37°$$

Angular speed of the connecting rod,

$$= \frac{\omega\cos\theta}{\left(n^2 - \sin^2\theta\right)^{0.5}}$$

$$= \frac{35.6 \times \cos 55.37°}{\left(25 - \sin^2 55.37°\right)^{0.5}} = 4.1 \text{ rad/s}$$

Acceleration of the piston,

$$\approx \omega^2 r\left[\frac{\cos\theta + \cos 2\theta}{n}\right]$$

$$= (35.6)^2 \times 0.12\left[\frac{\cos 55.37° + \cos 110.74°}{5}\right] = 75.65 \text{ m/s}^2$$

9.3 INERTIA FORCES IN THE RECIPROCATING ENGINE

9.3.1 Analytical Method

Consider the slider–crank chain, as shown in Fig.9.2.

Let W_r = weight of the connecting rod

R = weight of reciprocating parts

l = length of connecting rod

L = length of stroke = $2r$

r = radius of crank

l_1 = distance of centre of gravity G of connecting rod from the gudgeon pin(point P)

Total equivalent reciprocating weight.

$$R_e = \frac{R + (l - l_1)\,W_r}{l}$$

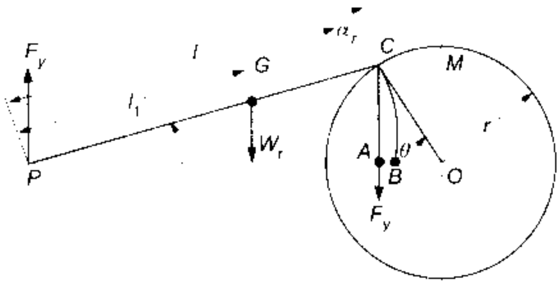


Fig.9.2 Inertia forces in reciprocating engine

The inertia force due to R_e.

$$F_i = -\frac{R_e a_p}{g} = -\left[R + \frac{(l - l_1)\,W_r}{l}\right]\frac{a_p}{g}$$

where a_p is the acceleration of the reciprocating parts

$$\approx \omega^2 r\left(\cos\theta + \frac{\cos 2\theta}{n}\right)$$

$$n = \frac{l}{r}$$

Torque exerted on the crankshaft due to inertia force.

$$T_i = F_i \cdot OM$$

where $$OM = \frac{r\sin(\theta + \phi)}{\cos\phi} \text{ and } \sin\phi = \frac{\sin\theta}{n}.$$

Correction couple,

$$T_o = W_r l_1 (l - L)\frac{\alpha_r}{g} \qquad (9.6)$$

where α_r is the angular acceleration of the rod

$$= \frac{-\omega^2\left(n^2 - 1\right)\sin\theta}{\left(n^2 - \sin^2 2\theta\right)^{1.5}} \approx \frac{-\omega^2\sin\theta}{n}$$

Let F_y be two equal and opposite forces applied at P and C.

Then $$F_y \cdot AP = T_o = W_r l_1 (l - L) \frac{\alpha_r}{g}$$

Corresponding torque on the crankshaft, $T_c = F_y \cdot AO$

or $$T_c = \{W_r l_1 (l - L) \alpha_r g\} \cdot (AO/AP)$$

Now $$AO = OC \cos\theta \quad \text{and} \quad AP = CP \cos\phi$$

Also $$\cos\phi = \left[1 - \sin^2\phi\right]^{0.5} = \left[n^2 - \sin^2\theta\right]^{0.5} / n$$

where n is $\dfrac{CP}{OC}$

$$\therefore \quad T_c = \left[W_r l_1 (l - L) \frac{\alpha_r}{g}\right] \cdot \left[\frac{\cos\theta}{\left(n^2 - \sin^2\theta\right)^{0.5}}\right]$$

or $$= -\left[\frac{W_r l_1 (l - L)}{g}\right] \cdot \left[\frac{\omega^2 \left(n^2 - 1\right) \sin 2\theta}{2\left(n^2 - \sin^2\theta\right)^2}\right]$$

$$\approx -\left[\frac{W_r l_1 (l - L)}{g}\right] \cdot \left[\frac{\omega^2 \sin 2\theta}{2n^2}\right] \tag{9.7}$$

Vertical force through $$C = W_r \cdot \frac{PG}{PC} = \frac{W_r l_1}{l}$$

Torque exerted on the crankshaft by gravity,

$$T_g = -\left(\frac{W_r l_1}{l}\right) \cdot AO$$

Now $$AO = OC \cos\theta = r\cos\theta$$

$$\therefore \quad T_g = -\left(\frac{W_r l_1}{n}\right) \cdot \cos\theta \tag{9.8}$$

Total torque exerted on the crankshaft by the inertia of moving parts

$$= T_i + T_c + T_g \tag{9.9}$$

9.3.2 Graphical Method

The graphical construction for calculating the inertia forces in a reciprocating engine is shown in Fig.9.3. The following procedure may be adopted for this purpose:

1. Draw the acceleration diagram OCQN by Klein's construction. The acceleration of the piston P with respect to the crank centre O is,

$$a_p = \omega^2 \cdot NO$$

and is acting in the direction from N to O. Therefore the inertia force F_i shall act in the opposite direction from O to N.

2. Replace the connecting rod by dynamically equivalent system of two masses as explained in Section 9.9. Let one of the masses be placed at P. To obtain the position of the other mass, draw GZ perpendicular to CP such that $GZ = K$, the radius of gyration of the connecting rod. Join PZ and from Z draw perpendicular to DZ which intersects CP at D. Now D is the position of the second mass. Otherwise, $GP \cdot GD = K^2$.

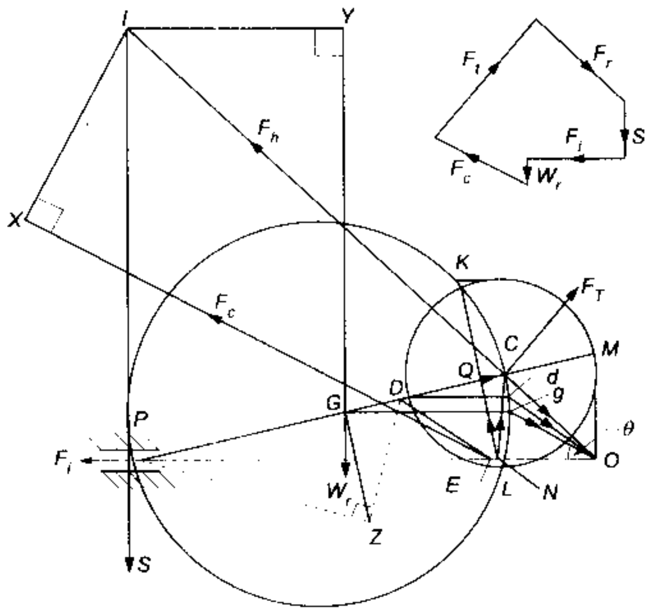


Fig.9.3 Inertia forces in reciprocating engines considering the weight of connecting rod

3. Locate the points g and d on NC, the acceleration image of the rod, by drawing parallel lines from G and D to the line of stroke. Join gO and dO. Then

$$a_G = \omega^2 \cdot gO \quad \text{and} \quad a_D = \omega^2 \cdot dO$$

4. From D, draw DE parallel to dO to intersect the line of stroke at E. The inertia force of the rod F_r acts through E and in the opposite direction,

$$F_r = m_r\omega^2 \cdot gO$$

where m_r is the mass of the rod.

The forces acting on the connecting rod are:

(i) Inertia force of the reciprocating parts F_i acting along the line of stroke PO.

(ii) The side thrust between the cross-head and the guide bars S acting at P and right angles to the line of stroke.

(iii) The weight of the connecting rod, $W_r = m_r g$.

(iv) Inertia force of the connecting rod F_r.

(v) The radial force F_r acting through O and parallel to the crank OC.

(vi) The force F_t acting perpendicular to the crank OC.

5. Now produce the line of action of F_r and S to intersect at a point I. From I draw IX and IY perpendicular to the lines of action of F_r and W_r. Taking moments about I, we have

$$F_t \cdot IC = F_i \cdot IP + F_r \cdot IX + W_r \cdot IY$$

The value of F_t may be obtained from this equation and from the force polygon, the forces S and F_r may be calculated. Then, torque exerted on the crankshaft to overcome the inertia of moving parts is $= F_t \cdot OC$.

9.4 EQUILIBRIUM OF FORCES IN SLIDER–CRANK CHAIN

Outstroke The various forces acting on the links of the slider–crank chain during outstroke are shown in Fig.9.4(a).

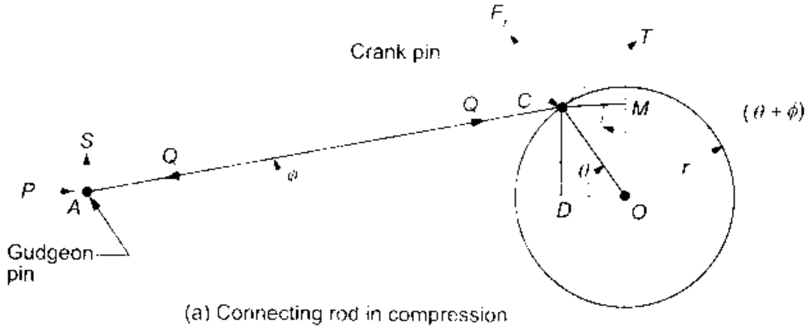
Let P = piston effort

Q = thrust in the connecting rod

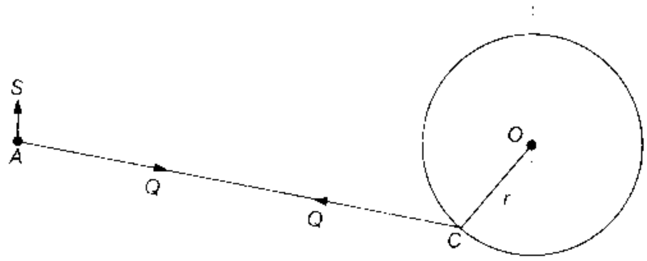
T = crank pin effort

W = force in the crank

S = reaction of guide bars



(a) Connecting rod in compression



(b) Connecting rod in tension

Fig.9.4 Turning moment in a reciprocating engine

In triangle *ACD*,
$$\frac{P}{\sin(90° - \phi)} = \frac{Q}{\sin 90°} = \frac{S}{\sin \phi}$$

or
$$\frac{P}{\cos \phi} = \frac{Q}{1} = \frac{S}{\sin \phi}$$

or
$$Q = \frac{P}{\cos \phi} \qquad (9.10)$$
$$S = P \tan \phi \qquad (9.11)$$

The connecting rod is in compression during outstroke.

$$T = Q \sin(\theta + \phi) \qquad (9.12)$$
$$W = Q \cos(\theta + \phi) \qquad (9.13)$$

Instroke The slider–crank chain during instroke is shown in Fig.9.4 (b). The connecting rod is under tension now. The various forces acting on the links are also shown in this figure.

Crank effort

Crank effort,

$$CE = T \cdot r = Qr\sin(\theta + \phi)$$

$$= Pr\frac{\sin(\theta + \phi)}{\cos\phi} = Pr[\sin\theta + \cos\theta\tan\phi]$$

Now

$$\sin\phi = \frac{\sin\theta}{n}$$

$$\tan\phi = \frac{\sin\theta}{\left(n^2 - \sin^2\theta\right)^{0.5}}$$

$$CE = Pr\left[\frac{\sin\theta + \sin 2\theta}{2\left(n^2 - \sin^2\theta\right)^{0.5}}\right] \tag{9.14}$$

$$= P \cdot OM$$

9.5 PISTON EFFORT

Double-acting horizontal steam engine The net force acting on the cross-head pin along the line of stroke is called piston effort.

Let D = diameter of the piston

d = diameter of the piston rod

p_1, p_4 = steam pressure on cylinder end during outstroke and instroke, respectively

p_2, p_3 = steam pressure on crank end during outstroke and instroke, respectively

R = weight of the reciprocating parts

Net force on piston during outstroke, $P_s = \left(\frac{\pi}{4}\right)\left[D^2 p_1 - \left(D^2 - d^2\right)p_2\right]$

Accelerating force due to reciprocating parts $= \left(\frac{R}{g}\right) \cdot \omega^2 r\left[\frac{\cos\theta + \cos 2\theta}{n}\right]$

Piston effort during outstroke,

$$PEO = \left(\frac{\pi}{4}\right)\left[D^2 p_1 - \left(D^2 - d^2\right)p_2\right] - \left(\frac{R}{g}\right) \cdot \omega^2 r\left[\frac{\cos\theta + \cos 2\theta}{n}\right] \tag{9.15}$$

Piston effort during instroke,

$$PEI = \left(\frac{\pi}{4}\right)\left[D^2 p_3 - \left(D^2 - d^2\right)p_4\right] - \left(\frac{R}{g}\right) \cdot \omega^2 r\left[\frac{\cos\theta + \cos 2\theta}{n}\right] \tag{9.16}$$

Double-acting vertical steam engine Piston effort during downstroke for vertical steam engine

= Piston effort for horizontal engine during downstroke + R (9.17)

Piston effort during upstroke for vertical steam engine

= Piston effort for horizontal engine during instroke − R (9.18)

Four stroke horizontal internal combustion engine Substitute p_2 and $p_3 = p_a$ = atmospheric pressure and $d = 0$ in (9.15) and (9.16) to obtain the piston effort during outstroke and instroke, respectively.

$$PEO = \left(\frac{\pi}{4}\right) D^2 (p_1 - p_a) - \left(\frac{R}{g}\right) \cdot \omega^2 r \left[\frac{\cos\theta + \cos 2\theta}{n}\right] \tag{9.19}$$

$$PEI = \left(\frac{\pi}{4}\right) D^2 (p_a - p_4) - \left(\frac{R}{g}\right) \cdot \omega^2 r \left[\frac{\cos\theta + \cos 2\theta}{n}\right] \tag{9.20}$$

Four stroke vertical internal combustion engine

$$PED = \left(\frac{\pi}{4}\right) D^2 (p_1 - p_a) - \left(\frac{R}{g}\right) \cdot \omega^2 r \left[\frac{\cos\theta + \cos 2\theta}{n}\right] + R \tag{9.21}$$

$$PEU = \left(\frac{\pi}{4}\right) D^2 (p_a - p_4) - \left(\frac{R}{g}\right) \cdot \omega^2 r \left[\frac{\cos\theta + \cos 2\theta}{n}\right] - R \tag{9.22}$$

Example 9.2

In a vertical double-acting steam engine running at 360 rpm, the cylinder diameter is 0.3 m, piston rod diameter is 40 mm and length of connecting rod is 0.7 m. When the crank has moved 120° from top dead centre, the pressure of steam at the cover end is 0.35 N/mm^2 and that at the crank end is 0.03 N/mm^2. If the weight of reciprocating parts is 500 N and length of stroke is 300 mm, find (a) piston effort and (b) turning moment on the crankshaft for the given crank position.

■ **Solution**

(a) Net force on the piston,

$$F_p = p_1 A_1 - p_2 A_2$$

$$= \left(\frac{\pi}{4}\right)\left[0.35 \times 300^2 - 0.03 \times 40^2\right] = 24702 \text{ N}$$

$$n = \frac{l}{r} = \frac{700}{150} = 4.67$$

Acceleration of the piston,

$$a_p = \omega^2 r \left[\frac{\cos\theta + \cos 2\theta}{n}\right]$$

$$= \left(\frac{2\pi \times 360}{60}\right)^2 \times 0.150 \times \left[\frac{\cos 120° + \cos 240°}{4.67}\right]$$

$$= -129.4 \text{ m/s}^2$$

Piston effort,

$$PE = F_p + R - \frac{Ra_p}{g}$$

$$= 24702 + 500 + \frac{500 \times 129.4}{9.81} = 31797 \text{ N}$$

(b) Turning moment on the crankshaft

$$= PE \cdot r \left[\sin\theta + \frac{\sin 2\theta}{2\left(n^2 - \sin^2\theta\right)^{0.5}}\right]$$

$$= 31797 \times 0.150 \left[\sin 120° + \frac{\sin 240°}{2\left(4.67^2 - \sin^2 120°\right)^{0.5}}\right]$$

$$= 3680.5 \text{ Nm}$$

Example 9.3

The radius of crank of a horizontal engine is 300 mm. The mass of the reciprocating parts is 200 kg. The difference between the driving and the back pressures is 0.4 N/mm^2, when the crank has travelled 60° from I.D.C. The length of connecting rod is 1.2 m and the cylinder bore is 0.5 m. The engine runs at 240 rpm. Neglecting the effect of the piston rod, find (a) pressure on the slide bar, (b) thrust in the connecting rod, (c) tangential force and (d) turning moment on the crankshaft.

■ **Solution**

$$\omega = \frac{2\pi \times 240}{60} = 25.13 \text{ rad/s}$$

$$n = \frac{l}{r} = \frac{1.2}{0.3} = 4$$

Force on the piston,

$$F_p = (p_1 - p_2) \times \frac{\pi D^2}{4}$$

$$= 0.40 \times \pi \times \frac{500^2}{4} = 78540 \text{ N}$$

Inertia force due to reciprocating parts,

$$F_i = \left(\frac{R}{g}\right) \omega^2 r \left[\cos\theta + \frac{\cos 2\theta}{n}\right]$$

$$= 200 \times (25.13)^2 \times 0.3 \left[\cos 60° + \frac{\cos 120°}{4}\right] = 14209 \text{ N}$$

Piston effort,

$$\text{PE} = F_p - F_i = 78540 - 14209 = 64331 \text{ N}$$

(a) Pressure on the slide bar:

$$\sin\phi = \frac{\sin\theta}{n} = \frac{\sin 60°}{4} = 0.2165$$

$$\phi = 12.5°$$

Pressure on the slide bars,

$$S = \text{PE} \tan\phi$$

$$= 64331 \tan 12.5° = 14266 \text{ N}$$

(b) Thrust in the connecting rod,

$$Q = \frac{\text{PE}}{\cos\phi}$$

$$= \frac{64331}{\cos 12.5°} = 65893 \text{ N}$$

(c) Tangential force on the crankpin,

$$T = Q \sin(\theta + \phi)$$

$$= 65893 \sin 72.5° = 62843 \text{ N}$$

(d) Turning moment on the crankshaft

$$= Tr$$

$$= 62843 \times 0.3 = 18853 \text{ Nm}$$

9.6 CRANK EFFORT (OR TURNING MOMENT) DIAGRAMS

The diagrams obtained on plotting crank effort for various positions of crank are known as crank effort diagrams.

Now $$x = r\left[(1 - \cos\theta) + \left\{n - \left(n^2 - \sin^2\theta\right)^{0.5}\right\}\right]$$

Length of stroke, $L = 2r$.

$$\therefore \quad \frac{x}{L} = 0.5\left[(1 - \cos\theta) + \left\{n - \left(n^2 - \sin^2\theta\right)^{0.5}\right\}\right] \tag{9.23}$$

x/L is called the *displacement constant*.

9.6.1 Procedure for Determination of Turning Moment Diagram

The following steps may be followed to determine the turning moment diagram:

1. Determine the pressure of the working fluid on both sides of the piston either from the indicator diagram or from theoretical calculations. Calculate the net load on the piston, P_s.
2. Calculate the acceleration of the piston, a_p.
3. Calculate the force due to reciprocating parts, $R \cdot \frac{a_p}{g}$.
4. Calculate the piston effort, $PE = P_s \pm R - R \cdot \frac{a_p}{g}$. Remember that second term is zero for a horizontal engine. For a vertical engine, take the positive sign before R for outstroke and the negative sign for instroke.
5. Calculate the crank effort, $CE = PE \times r\left[\sin\theta + \frac{\sin 2\theta}{2n}\right]$.

9.6.2 Turning Moment Diagram for a Vertical Steam Engine

The turning moment diagram for a vertical steam engine is shown in Fig.9.5. The hatched area below the mean torque shows deficient energy, and the area above the mean torque is the surplus energy. One cycle is completed during 360° of the crank rotation.

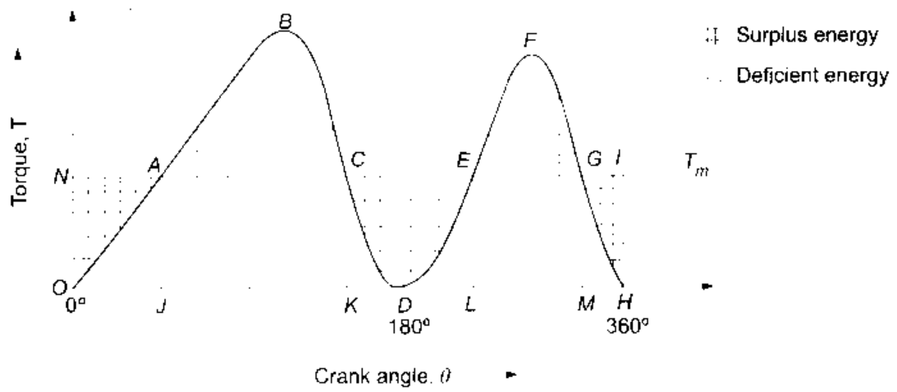


Fig.9.5 Turning moment diagram for a vertical steam engine

9.6.3 Turning Moment Diagram for a Four Stroke I.C. Engine

The turning moment diagram for a four stroke i.e. engine is shown in Fig.9.6. Here one cycle is completed during 720° of the crank rotation. Energy is supplied mainly during the expansion stroke. This diagram is more non-uniform as compared to Fig.9.5.

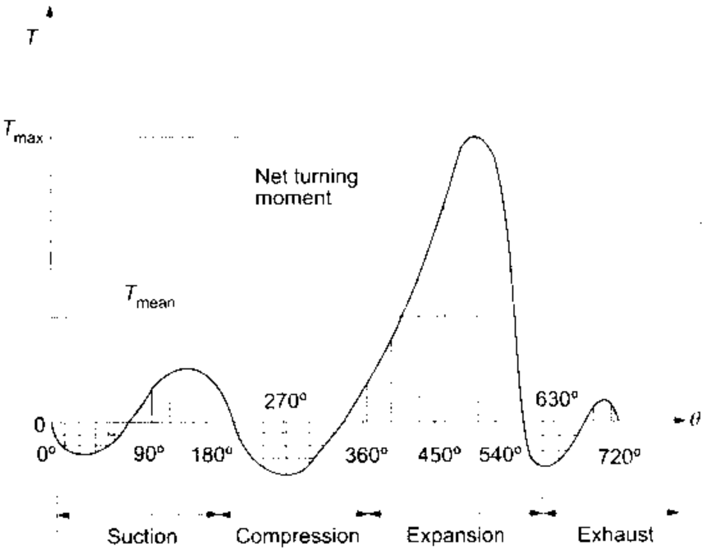


Fig.9.6 Turning moment diagram for a four stroke I.C. engine

9.6.4 Turning Moment Diagram for a Multi-cylinder Engine

The turning moment diagram for a three-cylinder engine is shown in Fig.9.7. The diagram becomes more uniform above and below the mean torque.

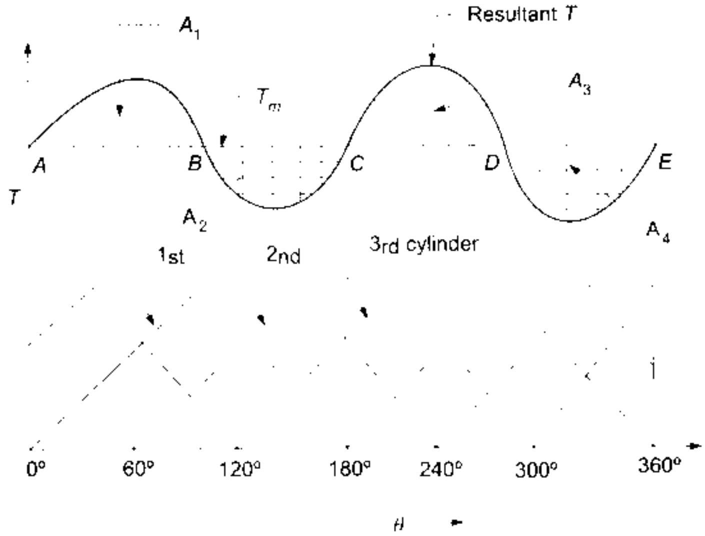


Fig.9.7 Turning moment diagram for a multi-cylinder engine

9.6.5 Uses of Turning Moment Diagram

The uses of turning moment diagram are:

1. The area under the turning moment diagram represents work done per cycle. This area multiplied by number of cycles per second gives the power developed by the engine in Watts.
2. By dividing the area of the turning moment diagram with the length of the base, we get the mean turning moment. This enables us to find the fluctuation of energy.
3. The maximum ordinate of the turning moment diagram gives the maximum torque to which the crankshaft is subjected to. This enables us to find the diameter of the crankshaft.

9.7 FLUCTUATION OF ENERGY

Fluctuation of energy (E_f) The fluctuation of energy is the excess energy developed by the engine between two crank positions.

$$E_f = K_e E \tag{9.24}$$

where E is 1/2. I ω_m^2. I = moment of inertia of the flywheel and ω_m its mean angular speed.

Coefficient of fluctuation of energy (K_e) The coefficient of fluctuation of energy is the ratio of the maximum fluctuation of energy to the indicated work done by the engine during one revolution of crank.

$$K_e = \frac{E_{\max} - E_{\min}}{T_m \cdot \theta} \tag{9.25}$$

$\theta = 4\pi$ for steam engines and 4π for four stroke internal combustion engines. Mean torque,

$$T_m = \frac{\text{Power developed}}{\omega_m}$$

Coefficient of fluctuation of speed (K_s) The coefficient of fluctuation of speed is defined as the ratio of the difference between the maximum and minimum angular velocities of the crankshaft to its mean angular velocity.

$$K_s = \frac{\omega_{\max} - \omega_{\min}}{\omega_m} = \frac{N_{\max} - N_{\min}}{N_m} \tag{9.26}$$

where

$$N_m = \frac{N_{\max} + N_{\min}}{2}$$

9.7.1 Determination of Maximum Fluctuation of Energy

As shown in Fig.9.7, let E be the energy at point A.

Then Energy at $B = E + A_1$

Energy at $C = E + A_1 - A_2$

Energy at $D = E + A_1 - A_2 + A_3$

Energy at $E = E + A_1 - A_2 + A_3 - A_4$ = Energy at A, that is E

Let the maximum energy be at point B and minimum at point D. Then

$$E_{max} = E + A_1$$
$$E_{min} = E + A_1 - A_2 + A_3$$

Maximum fluctuation of energy $= E_{max} - E_{min}$

$$= A_2 - A_3$$

9.8 FLYWHEEL

A flywheel is a device which serves as a reservoir to store energy when the supply of energy is more than the requirement and releases energy when the requirement is more than the supply. Thereby, it controls the fluctuation of speed of the prime mover during each cycle of operation. The differences between the functions of a flywheel and governor are given in Table 9.1.

Table 9.1 Differences between the functions of a flywheel and a governor

Flywheel	Governor
1. It decreases the variation of speed of the prime mover during each cycle of operation	1. It regulates the speed of the prime mover from cycle to cycle.
2. It decreases the fluctuation of speed due to difference in output and input.	2. It decreases the fluctuation of speed by adjusting the output of the prime mover.
3. A flywheel controls dN/dt.	3. A governor controls dN.
4. It stores energy and gives up when required.	4. It regulates speed by regulating the quantity of working medium of the prime mover.
5. It has no control over the quality of the working medium.	5. It takes care of the quality of the working medium.
6. It is not an essential part of every prime mover.	6. It is an essential part of every prime mover.

9.8.1 Size of Flywheel

Let N_{max} = maximum speed of the flywheel in rpm

N_{min} = minimum speed of the flywheel in rpm

I = moment of inertia of the flywheel about a polar axis

$= (W/g) \cdot K^2$

W = weight of the flywheel

K = radius of gyration of the flywheel

Maximum kinetic energy of the flywheel, $E_{max} = \left(\frac{1}{2}\right) \cdot I\omega_{max}^2$

$$= \left(\frac{1}{2}\right) \cdot \left(\frac{W}{g}\right) K^2 \left(\frac{2\pi N_{max}}{60}\right)^2$$

Minimum kinetic energy of the flywheel. $E_{min} = \left(\frac{1}{2}\right) \cdot I\omega_{min}^2$

$$= \left(\frac{1}{2}\right) \cdot \left(\frac{W}{g}\right) K^2 \left(\frac{2\pi N_{min}}{60}\right)^2$$

Fluctuation of energy, $E_f = E_{max} - E_{min}$

$$= \left(\frac{1}{2}\right) \cdot \left(\frac{W}{g}\right) K^2 \left(\frac{2\pi}{60}\right)^2 \left(N_{max}^2 - N_{min}^2\right)$$

$$= \left(\frac{\pi^2}{1800}\right) \cdot \left(\frac{WK^2}{g}\right) \cdot (N_{max} + N_{min})(N_{max} - N_{min})$$

$$= \left(\frac{\pi^2}{1800}\right) \cdot \left(\frac{WK^2}{g}\right) \cdot (2N_m \times N_m K_s)$$

or

$$W = \frac{900 \times g \times E_f}{\pi^2 \times K^2 \times N_m^2 \times K_s} \tag{9.27}$$

There are two type of flywheels: disc type and arm type. In the arm type of flywheel, the weight of the flywheel is mainly located in the rim and the arms and boss do not contribute much in storing the energy. The hoop stress in the flywheel can be determined by assuming it as a ring.

Hoop stress,

$$\sigma_h = \rho v^2 \tag{9.28}$$

where ρ is the density of the rim and v is its peripheral speed.

If b is the width of the rim and t its thickness, then

$$W = \rho \cdot \pi d \cdot bt \cdot g \tag{9.29}$$

where d is the diameter of the flywheel.

9.8.2 Flywheel for a Punching Press

The mechanism used for punching of holes in plates is shown in Fig.9.8.

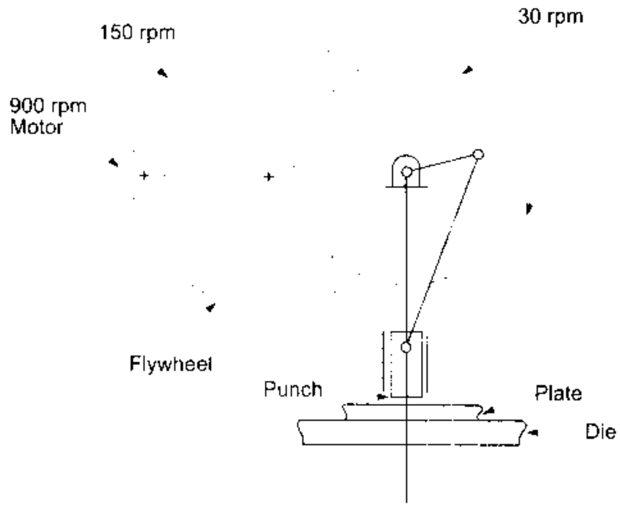


Fig.9.8 Punching press

It consists of a driving motor whose speed is reduced to the desired value by a set of gears to drive the slider–crank mechanism. The slider of the slider–crank is used as the punch to punch a hole in the plate placed on the die. The punching takes place for a very short interval of the angle of rotation of the crank. The punching operation requires huge amount of energy for a short time. For the remaining period the device remains idle. Therefore, the flywheel is used to store energy during the idle period and supply the desired energy during the working period. The distribution of force on the punch during the punching operation is shown in Fig.9.9.

Let d = diameter of hole punched

t = thickness of plate

τ_u = ultimate shear strength of plate material

Maximum shear force required during punching,

$$F = \pi d t \tau_u$$

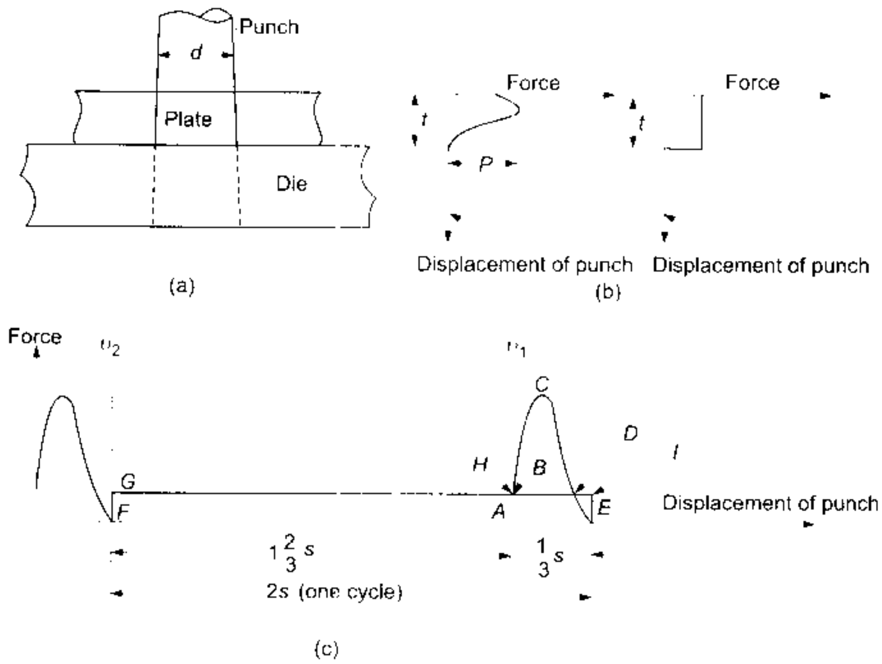


Fig.9.9 Distribution of forces during punching

Energy required for punching a hole, $E_1 = 0.5Ft$

Let θ = angle of crank during which punching takes place

Energy supplied by the motor to the crankshaft during punching operation,

$$E_2 = \frac{E_1\theta}{2\pi}$$

Balance energy required for punching,

$$E_f = E_1 - E_2 = E_1\left(1 - \frac{\theta}{2\pi}\right) \tag{9.30}$$

It may be assumed that, $$\frac{\theta}{2\pi} = \frac{t}{2L} = \frac{t}{4r}$$

where L is the stroke length and r the radius of crank.

The balance of energy is to be supplied by the flywheel.

$$E_f = 0.5M\left(v_1^2 - v_2^2\right) = MK_s v_m^2 \quad (9.31)$$

Example 9.4

The turning moment diagram for one revolution of a multi-cylinder engine is shown in Fig.9.10. The vertical and horizontal scales are:

1 mm = 600 Nm and 2.5° respectively. The fluctuation of speed is limited to ±1.5% of mean speed, which is 250 rpm. The hoop stress in rim material is limited to 5.6 N/mm². Neglecting effect of boss and arms, determine the suitable diameter and cross-section of flywheel rim. Take density of rim material as 7200 kg/m³ and width to be four times the thickness.

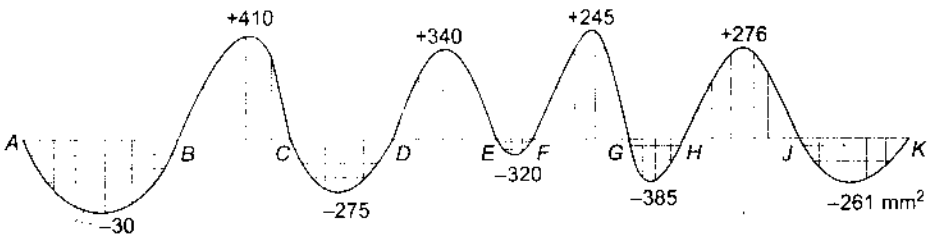


Fig.9.10 Turning moment diagram of multi-cylinder engine

■ **Solution**

Hoop stress,
$$\sigma_H = \rho v^2$$
$$v = \pi d N_m \frac{K_s}{60}$$
$$= \pi d \times 250 \times \frac{1.015}{60} = 13.286\, d \text{ m/s}$$
$$5.6 \times 10^6 = 7200 \times (13.286\, d)^2$$
$$d = 2.1 \text{ m}$$

Let E be the energy at A.

Fluctuation of energy is shown in Table 9.2

$$E_f = E_{max} - E_{min} = (E + 445) - (E - 30)$$
$$= 475 \text{ mm}^2$$
$$= 475 \times 10^{-6} \times \left(\frac{2.5 \times \pi \times 10^3}{180}\right) \times 600 \times 10^3$$
$$= 12435.47 \text{ Nm}$$
$$W = \frac{900 \times g \times E_f}{\pi^2 \times K^2 \times N_m^2 \times K_s}$$

$$= \frac{900 \times 9.81 \times 12435.47}{\pi^2 \times 1.05^2 \times 250^2 \times 0.03} = 5381.4 \text{ N}$$

Now $W = \pi d b t \rho g$

$5381.4 = \pi \times 2.1 \times 4t \times t \times 7200 \times 9.81$

$t = 53.73$ mm and $b = 214.92$ mm

Table 9.2

Point	Energy (mm^2)
B	$E - 30$
C	$E - 30 + 410 = E + 380$
D	$E + 380 - 275 = E + 105$
E	$E + 105 + 340 = E + 445$
F	$E + 445 - 320 = E + 125$
G	$E + 125 + 245 = E + 370$
H	$E + 370 - 385 = E - 15$
J	$E - 15 + 276 = E + 261$
K	$E + 261 - 261 = E$

Example 9.5

The variation of torque for an intermittent operation of a machine is shown in Fig.9.11. The machine is directly coupled to a motor which exerts a constant torque at a mean speed of 200 rpm. The flywheel has a moment of inertia of 2000 kg m^2. Determine (a) the mean power of the motor, and (b) total fluctuation of speed of machine shaft.

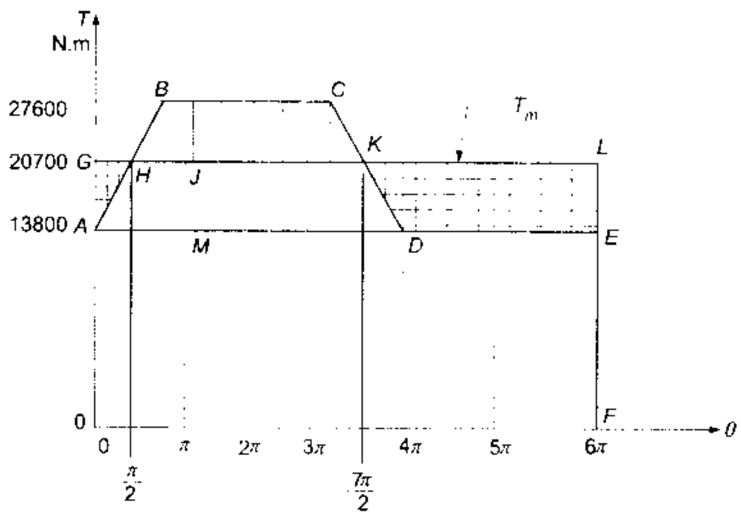


Fig.9.11 Variation of torque for an intermittent operation

Solution

$$\text{Area of turning moment diagram} = \text{Area } OAEF + \text{Area } ABCD$$

$$= 13800\left[6\pi + \frac{4\pi + 2\pi}{2}\right] = 13800 \times 9\pi$$

$$= T_m \times 6\pi$$

or $T_m = 20700$ Nm

Power developed, $P = \dfrac{2\pi N_m T_m}{60 \times 1000}$ kW

$$= \frac{2\pi \times 200 \times 20700}{60 \times 1000} = 433.54 \text{ kW}$$

Surplus energy is represented by the area *HBCK* and deficient energy by the areas *AGH* and *KLED*.

Now $\dfrac{GH}{HJ} = \dfrac{GA}{BJ}$

or $\dfrac{GH}{GH + HJ} = \dfrac{GA}{GA + BJ}$

$$\frac{GH}{GJ} = \frac{GA}{BM}$$

$$GH = \frac{\pi \times 6900}{13800} = \frac{\pi}{2}$$

Therefore, $\theta_H = \dfrac{\pi}{2}$

Similarly $\theta_K = \dfrac{7\pi}{2}$

Surplus energy $= \text{Area } HBCK$

$$= \left[\frac{3\pi + 2\pi}{2}\right](27600 - 20700) = 54192 \text{ Nm}$$

$$= \left(\frac{\pi^2}{1800}\right) \cdot I \cdot (N_{max} + N_{min})\,(N_{max} - N_{min})$$

$$= \left(\frac{\pi^2}{1800}\right) \times 2000 \times (200 \times 2)\,(N_{max} - N_{min})$$

$$N_{max} - N_{min} = 12.35 \text{ rpm}$$

Percentage fluctuation $= \dfrac{12.35 \times 100}{200} = 6.18\%$

Example 9.6

A certain machine requires a torque of $(500 + 50 \sin\theta)$ Nm to drive it, where θ is the angle of rotation of shaft measured from a certain datum. The machine is directly coupled to an engine which produces a torque of $(500 + 60 \sin 2\theta)$ Nm. The flywheel and the other rotating parts attached to the engine weigh 500 N and have a radius of gyration of 0.4 m. The mean speed is 180 rpm. Determine (a) The fluctuation of energy, (b) the percentage fluctuation of speed, and (c) the maximum and minimum angular acceleration of the flywheel and corresponding shaft positions.

Solution

(a) Change in torque $= (500 + 60\sin 2\theta) - (500 + 50\sin\theta)$

$$= 60\sin 2\theta - 50\sin\theta$$
$$= 120\sin\theta\cos\theta - 50\sin\theta$$
$$= \sin\theta(120\cos\theta - 50)$$

For change in torque to be zero, $\sin\theta = 0$, or $\theta = 0°$, $180°$ and $360°$.

Also, $\cos\theta = \frac{50}{120} = 0.4167$ or $\theta = 65.4°$ and $294.6°$

The variation of T against θ is shown in Fig.9.12.

Fluctuation of energy, $E_f = \int_{180°}^{294.6°} (60\sin 2\theta - 50\sin\theta)d\theta$

$$= [-30\cos 2\theta + 50\cos\theta]_{180°}^{294.6°} = 121 \text{ Nm}$$

(b)
$$E_f = \left(\frac{\pi^2}{1800}\right)\cdot\left(\frac{WK^2}{g}\right)\cdot(N_{max} + N_{min})(N_{max} - N_{min})$$
$$121 = \left(\frac{\pi^2}{1800}\right)\cdot\left(\frac{500 \times 0.4^2}{9.81}\right)\cdot(180 \times 2)(N_{max} - N_{min})$$
$$N_{max} - N_{min} = 7.45 \text{ rpm}$$

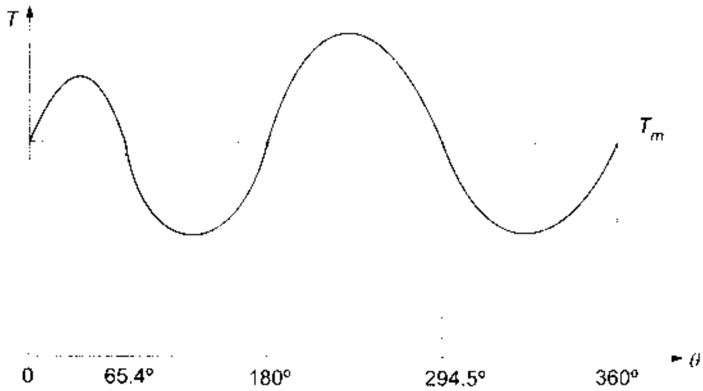


Fig.9.12 Variation of τ against θ

Percentage fluctuation of speed $= \frac{7.45 \times 100}{180} = 4.14\%$.

(c) $T = 60\sin 2\theta - 50\sin\theta$

For maximum or minimum value of T,

$$\frac{dT}{d\theta} = 120\cos 2\theta - 50\cos\theta = 0$$

or $$120\left(2\cos^2\theta - 1\right) - 50\cos\theta = 0$$

or $$240\cos^2\theta - 50\cos\theta - 120 = 0$$

or $$24\cos^2\theta - 5\cos\theta - 12 = 0$$

$$\cos\theta = \frac{5 \pm (25 + 1152)^{0.5}}{48}$$

$$= 0.8189 \quad \text{and} \quad -0.61057$$

$$\theta = 35^\circ \quad \text{and} \quad 127.6^\circ$$

$$T_{max} = 60\sin 70^\circ - 50\sin 35^\circ = 27.7 \text{ Nm}$$

Maximum acceleration. $$\alpha_{max} = \frac{T_{max}g}{WK^2}$$

$$= \frac{27.7 \times 9.81}{500 \times 0.16} = 3.397 \text{ rad/s}^2$$

$$T_{min} = 60\sin 255.2^\circ - 50\sin 127.6^\circ = -97.62 \text{ Nm}$$

Minimum acceleration. $$\alpha_{min} = \frac{97.62 \times 9.81}{500 \times 0.16} = 11.97 \text{ rad/s}^2$$

Example 9.7

A single cylinder, single acting, four stroke gas engine develops 25 kW at 320 rpm. The work done by the gases during the expansion stroke is three times the work done on the gases during the compression stroke. The work done during the suction and exhaust strokes being negligible. The fluctuation of speed is not to exceed ±2% of the mean speed. The turning moment diagram during compression and expansion is assumed to be triangular in shape. Find the weight of the flywheel if its radius of gyration is 0.5 m.

Solution

Coefficient of fluctuation of speed, $K_s = 4\%$

$$\text{Work done per cycle} = \frac{60P}{n}$$

$$= \frac{60 \times 25000}{160} = 9375 \text{ Nm}$$

Net work done per cycle = Work done during expansion – Work done during compression

$$= W_e - W_c = \frac{W_e - W_e}{3}$$

$$= \frac{2W_e}{3} = 9375$$

or $$W_e = 14062.5 \text{ Nm}$$

The $T - \theta$ diagram is shown in Fig.9.13.

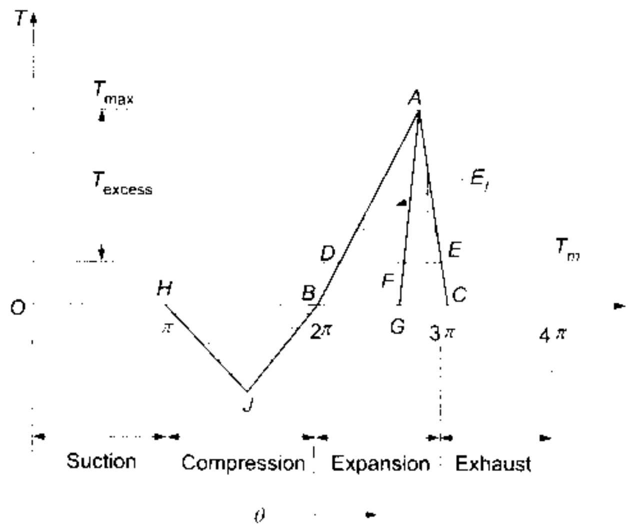


Fig.9.13 Variation of T against θ

$$\text{Work done during expansion stroke} = \text{area } ABC = 0.5 \times BC \times AG$$

$$14062.5 = 0.5 \times \pi \times AG$$

or

$$AG = 8952.5 \text{ Nm} = T_{max}$$

Mean turning moment,

$$T_m = FG = \frac{9375}{4\pi} = 746 \text{ Nm}$$

Excess turning moment,

$$T_{excess} = AF = AG - FG$$

$$= 8952.5 - 746 = 8206.5 \text{ Nm}$$

From similar triangles ADE and ABC, we have

$$\frac{DE}{BC} = \frac{AF}{AG}$$

or

$$DE = \frac{AF}{AG} \cdot BC = \left(\frac{8206.5}{8952.5}\right)\pi = 2.88 \text{ rad}$$

Maximum fluctuation of energy,

$$E_f = \text{area ADE} = 0.5 \cdot DE \cdot AF$$

$$= 0.5 \times 2.88 \times 8206.5 = 11817.4 \text{ Nm}$$

Now

$$E_f = K_s\left(\frac{W}{g}\right)K^2\omega^2$$

$$11817.4 = 0.04\left(\frac{W}{9.81}\right) \times (0.5)^2 \times \left(\frac{2\pi \times 320}{60}\right)^2$$

$$W = 10324 \text{ N}$$

Example 9.8

An Otto cycle engine develops 45 kW at 180 rpm with 90 explosions per minute. The change of speed from the commencement to the end of power stroke must not exceed 0.5% of mean on either side. Find the mean diameter of the flywheel and rim cross-section having width four times the thickness so that the hoop stress does not exceed 3.5 MPa. Assume that the flywheel stores 6% more energy than the energy stored by the rim and the work done during power stroke is 1.4 times the work done during the cycle. Take density of rim material to be 7300 kg/m^3.

■ Solution

$$\omega = \frac{2\pi \times 180}{60} = 18.85 \text{ rad/s}$$

Power developed, $P = T_m\omega$

or $T_m = \dfrac{45000}{18.85} = 2387.3$ Nm

Since the number of explosions are half of the rpm, therefore, it is a four stroke engine. The turning moment diagram is shown in Fig.9.14.

Work done per cycle $= T_m\theta = 2387.3 \times 4\pi = 30000$ Nm.

Work done during the power stroke $= 1.4 \times 30000 = 42000$ Nm.

Triangle *ABC* shows the work done during the power stroke.

Work done during the working stroke $=$ area ABC $= 0.5 \cdot AC \cdot BF$

$$42000 = 0.5T_{max} \times \pi$$

or $T_{max} = 26738$ Nm

Excess torque, $T_{excess} = BG = BF - GF = T_{max} - T_m$

$$= 26738 - 2387.3 = 24350.7 \text{ Nm}$$

From similar triangles *ABC* and *BDE*, we have

$$\frac{DE}{AC} = \frac{BG}{BF}$$

or

$$DE = \left(\frac{BG}{BF}\right) \cdot AC = \frac{T_{excess}}{T_{max}} \times \pi$$

$$= \left(\frac{24350.7}{26738}\right)\pi = 0.9107\pi$$

Maximum fluctuation of energy,

$$E_f = \text{area BDE} = 0.5DE \cdot BG$$

$$= 0.5 \times 0.9107\pi \times 24350.7$$

$$= 34834 \text{ Nm}$$

Hoop stress in flywheel rim $= \rho v^2$

$$3.5 \times 10^6 = 7300v^2$$

or $v = 21.896$ m/s

Let d be the diameter of flywheel.

$$21.896 = \pi d \times \frac{180}{60}$$

or $d = 2.32$ m

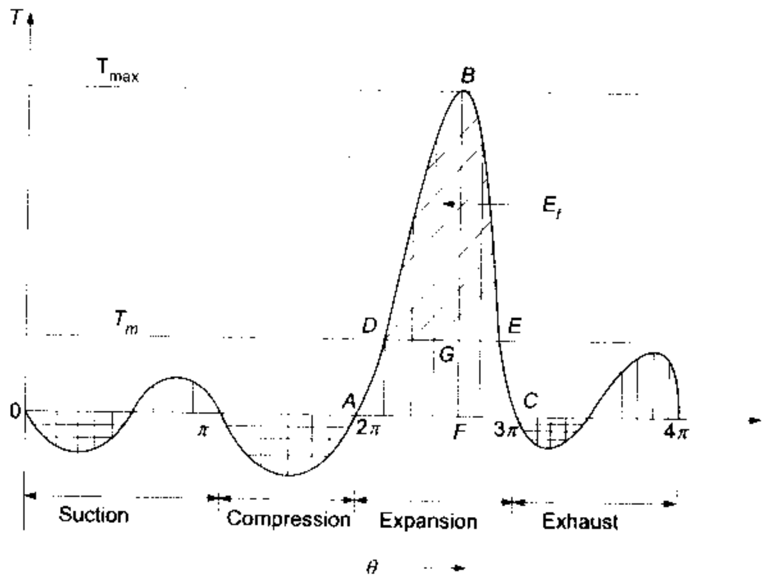


Fig.9.14 Variation of T against θ

Fluctuation of speed = 1%

Coefficient of fluctuation of speed, $K_s = 0.01$

Now $\quad E_f = 2EK_s$

$$34834 = 2 \times 0.01 \times E$$

$$E = 1741.7 \times 10^3 \text{ Nm}$$

Energy stored by the flywheel $= 1.06E = 1846.2 \times 10^3$ Nm

$$= 0.5\left(\frac{W}{g}\right)v^2$$

$$= 0.5 \times W \times \frac{(21.896)^2}{9.81}$$

or $\quad W = 75552 \text{ N} = \pi dbt\rho g$

$$= \pi \times 2.32 \times 4t^2 \times 7300 \times 9.81$$

or $\quad t = 190$ mm

$$b = 760 \text{ mm}$$

Example 9.9

A punching press is required to punch 30 holes per minute of 20 mm diameter in a steel plate 13 mm thick. The actual punching takes place at 1/6th of the interval between punches. The shear strength of the plate is 310 N/mm^2. The driving motor runs at 900 rpm with a reduction in the velocity through gears to give the desired speed of 30 punching operations per minute. Find the mass of the flywheel required if its mean diameter is 900 mm. Take K_s = 10%.

■ **Solution**

Required punching force, $\quad F = \pi d t \tau_u$

$$= \pi \times 20 \times 13 \times 310 = 253212 \text{ N}$$

The force versus displacement diagram is shown in Fig.9.9. The area under the curve can be approximated as a triangle, so that the work done in punching the hole is,

$$= 0.5Ft$$
$$= 0.5 \times 253212 \times 0.013 = 1645.9 \text{ Nm}$$

Time between punching operations $= \dfrac{60}{30} = 2$ s

Punching time $= \dfrac{2}{6} = \dfrac{1}{3}$ s

Average power required without the flywheel $= \dfrac{1645.9}{(1/3)} = 4937.6$ Watts

Since F in Fig.9.9(a) is twice as large as in Fig.9.9(b), therefore the instantaneous power required will be 9875 watts.

When a flywheel is used, the force-displacement curve is shown in Fig.9.9(c). The work required to punch the hole is represented by the area *ABCDE*. The same amount of energy to be supplied by the flywheel is represented by the area *FGIE*. Therefore, 1645.9 Nm of energy is to be supplied in 2 s, that is, a 822.95 watt motor is required. During the 1/3 second punching interval, the motor supplies the energy represented by the area *AHIE*, which is 822.95/3 = 274.3 Nm. But the energy required is 1645.9 Nm. Therefore, the energy to be taken from the flywheel is, E_f = 1645.9 − 274.3 = 1371.6 Nm.

Mean velocity, $v_m = \pi \times 0.9 \times \dfrac{150}{60} = 7.07$ m/s

$$v_1 + v_2 = 2v_m = 14.14$$
$$v_1 - v_2 = K_s v_m = 0.1 \times 7.07 = 0.707$$
$$v_1 = 7.424 \text{ m/s} \quad \text{and} \quad v_2 = 6.717 \text{ m/s}$$

Mass of flywheel required, $M = \dfrac{E_f}{K_s \cdot v_m^2} = \dfrac{1371.6}{0.1 \times 7.07^2} = 274.4$ kg

Example 9.10

A punching machine is required to punch 5 holes per minute of 50 mm diameter in 40 mm thick plate. The ultimate shear strength of plate material is 225 MPa. The punch has a stroke of 100 mm. Find the power of motor required if mean speed of flywheel is 18 m/s. If coefficient of fluctuation of energy is 4%, find the mass of the flywheel.

■ Solution

Punching force, $F = \pi d t \tau_u = \pi \times 50 \times 40 \times 225 = 1413717$ N

Punching time per hole $= \dfrac{60}{5} = 12$ s

Energy required in punching one hole, $E_1 = 0.5\, FK_e$

$$= 0.5 \times 1413717 \times 0.04 = 28274 \text{ Nm}$$

Power required = E_1 / punching time $= \dfrac{28274}{12} = 2356$ Watt or 2.356 kW

The punch travels a total distance of 2 × 100 = 200 mm (upstroke + downstroke) in 12 s.

Time required to punch a hole in 40 mm thick plate $= \dfrac{12 \times 40}{200} = 2.4$ s

Enrgy required to be supplied by motor in 12 s = 28274 Nm

Energy supplied by the motor in 2.4 s = $28274 \times \frac{2.4}{12} = 5654.8$ Nm

Energy supplied by flywheel. $E_f = 28274 - 5654.8 = 22619.8$ Nm

$$E = \frac{E_f}{2K_e} = \frac{22619.2}{2 \times 0.04} = 282740 \text{ Nm}$$

If M is the mass of the flywheel, then

$$0.5Mv^2 = E$$

$$M = 2 \times \frac{282740}{(18)^2} = 1745 \text{ kg}$$

Example 9.11

A punching press makes 25 holes of 20 mm diameter per minute in a plate 15 mm thick. This causes variation in the speed of flywheel attached to press from 240 to 220 rpm. The punching operation takes 2 seconds per hole. Assuming 6 Nm of work is required to shear 1 mm^2 of the area and frictional losses account for 15% of the work supplied for punching, determine (a) the power required to operate the punching press, and (b) the mass of flywheel with radius of gyration of 0.5 m.

■ **Solution**

Work required for punching one hole = Area of shear in mm^2 × Work per mm^2

$$= \pi dt \times 6 = \pi \times 20 \times 15 \times 6$$

$$= 5654.86 \text{ Nm}$$

Accounting 15% for frictional losses, the actual work supplied

$$= \frac{5654.86}{0.85} = 6652.78 \text{ Nm}$$

Total work required per minute for drilling 25 holes

$$= 6652.78 \times 25 = 166319 \text{ Nm}$$

(a) Power required $= \frac{166319}{60 \times 10^3} = 2.772$ kW

Energy supplied during the punching operation = $2.772 \times 1000 \times 2 = 5544$ Nm

Energy supplied by the flywheel, $E_f = 6652.78 - 5544 = 1108.78$ Nm

$$E_f = 0.5I\left(\omega_1^2 - \omega_2^2\right)$$

$$1108.78 = 0.5M \times (0.5)^2 \times \left(\frac{2\pi}{60}\right)^2 \left[240^2 - 220^2\right]$$

$$M = 87.92 \text{ kg}$$

Example 9.12

An electric motor drives a punching press to which a flywheel of radius of gyration 0.5 m is fitted. The flywheel runs at 240 rpm. The press is capable of punching 600 holes per hour with each punching operation taking 2 seconds and requiring 15 kNm of work. Determine (a) the rating of the motor, and (b) mass of the flywheel if its speed does not drop below 220 rpm.

Solution

(a) Total work required per hour $=$ Work per hole $\times$ Number of holes per hour

$= 15 \times 600 = 9 \times 10^6$ Nm

$$\text{Motor power} = \frac{9 \times 10^6}{10^3 \times 3600} = 2.5 \text{ kW}$$

(b) Energy delivered by motor during the punching operation.

$$E_2 = 2.5 \times 1000 \times 2 = 5000 \text{ Nm}$$

Energy required per punch operation, $E_1 = 15000$ Nm

Fluctuation of energy, $E_f = E_1 - E_2 = 15000 - 5000 = 10000$ Nm

$$E_f = 0.5I\left(\omega_1^2 - \omega_2^2\right)$$

$$10000 = 0.5M \times (0.5)^2 \times \left(\frac{2\pi}{60}\right)^2 \left[240^2 - 220^2\right]$$

$$M = 792.95 \text{ kg}$$

Example 9.13

A 5 kW induction motor running at 750 rpm operates a rivetting machine. A flywheel of mass 80 kg and radius of gyration 0.45 m is fitted to it. Each rivetting takes 1 s and requires 10 kW. Determine (a) number of rivets closed per hour, and (b) fall in speed of the flywheel after the riveting operation.

Solution

Mean speed, $\omega_m = 2\pi \times \frac{750}{60} = 78.54$ rad/s

Energy supplied by the motor in one hour $= 5 \times 10^3 \times 3600 = 18 \times 10^6$ Nm

Energy required for one riveting operation $= 10 \times 10^3 \times 1 = 10^4$ Nm

Number of rivets closed per hour $= \frac{18 \times 10^6}{10^4} = 1800$

Energy supplied by the motor in 1 s $= 5 \times 10^3 \times 1 = 5000$ Nm

Energy to be supplied by the flywheel $= 10000 - 5000 = 5000$ Nm

$$E_f = E_{max} - E_{min} = 0.5I\left(\omega_{max}^2 - \omega_{min}^2\right)$$

Now $\omega_{max} = \omega_m$

$$5000 = 0.5 \times 80 \times (0.45)^2 \left[(78.540^2 - \omega_{min}^2\right]$$

$$\omega_{min} = 74.5 \text{ rad/s}$$

or $N_{min} = 711.5$ rpm

Fall in speed $= 750 - 711.5 = 38.5$ rpm

9.9 EQUIVALENT DYNAMICAL SYSTEM

A continuous body may be replaced by a body by two masses assumed to be concentrated at two points and connected rigidly together. Such a system of two masses is termed an equivalent dynamical system. The conditions to be satisfied by an equivalent dynamical system are:

1. The total mass must be equal to that of the rigid body.
2. The centre of gravity must coincide with that of the rigid body.
3. The total moment of inertia about an axis through centre of gravity must be equal to that of the rigid body.

Consider a rigid body as shown in Fig.9.15.

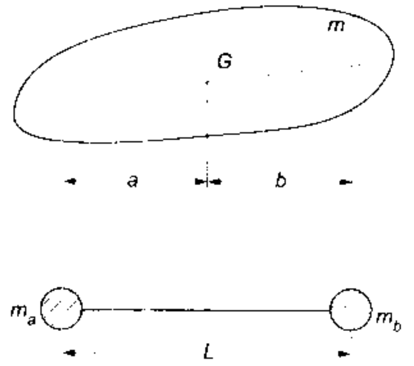


Fig.9.15 Truely dynamical system

Let m = mass of the rigid body

K = radius of gyration about an axis through G

m_a, m_b = two masses for equivalent dynamical system

a, b = distances of m_a, m_b from G respectively.

Then
$$m_a + m_b = m \tag{9.32}$$
$$m_a \cdot a = m_b \cdot b \tag{9.33}$$
$$m_a \cdot a^2 + m_b \cdot b^2 = mK^2 \tag{9.34}$$

Solving (9.33) and (9.34), we have

$$m_a \cdot a^2 + m_a \cdot ab = mK^2$$

or
$$m_a = \frac{mK^2}{a(a+b)} \tag{9.35}$$

From (9.32) and (9.33), we get

$$m_a = \frac{mb}{a+b} \tag{9.36}$$

Comparing (9.35) and (9.36), we get

$$K^2 = ab \tag{9.37}$$

Let L = length of a simple pendulum which has the same period of oscillations as the body with length equal to $(a + b)$.

Therefore the second mass is situated at the centre of percussion of the body.

A compound pendulum equivalent to the rigid body is shown in Fig.9.16. The centre of oscillation is A and the centre of percussion is at A'.

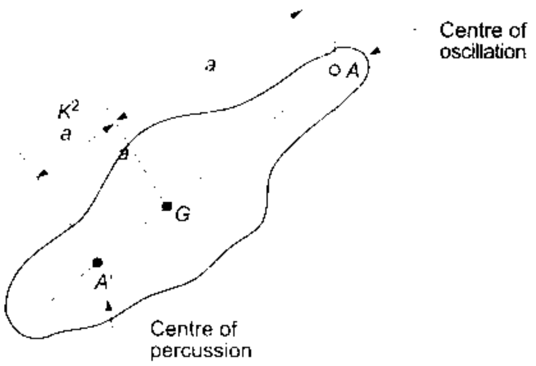


Fig.9.16 Compound pendulum

$$\text{Therefore} \quad L = \frac{K^2}{a} + a = a + b$$

$$\text{or} \quad b = \frac{K^2}{a}$$

$$\text{or} \quad K^2 = ab$$

In an approximate dynamical system, as shown in Fig.9.17, the distances a and c are fixed arbitrarily, then

$$m_a = \left[\frac{c}{a + c}\right] m$$

$$m_c = \left[\frac{a}{a + c}\right] m$$

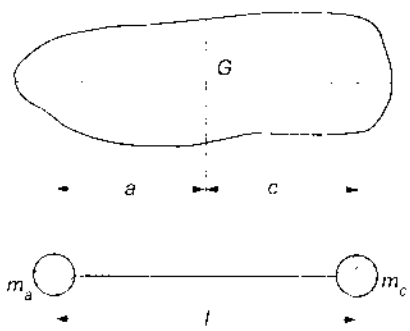


Fig.9.17 Approximate dynamical system

Mass moment of inertia of m_a and m_c about G is

$$I_1 = \left[\frac{m}{a+c}\right]\left(a^2c + c^2a\right)$$
$$= m \cdot ac = m_a a^2 + m_c c^2$$

Let $\quad K_1$ = radius of gyration of two-mass system.

Then $\quad I_1 = mK_1^2$

Therefore $\quad K_1^2 = ac \qquad (9.38)$

$$\text{Difference in mass moment of inertia} = I_1 - I$$
$$= m\left(K_1^2 - K^2\right)$$

Let α be the angular acceleration of the body.
Difference in torque or correction couple.

$$T_o = m\left(K_1^2 - K^2\right) \cdot \alpha$$

Let $\quad l$ = distance between two masses m_a and m_c fixed arbitrarily

L = distance between two masses m_a and m_b which form a true dynamically equivalent system.

Then
$$c - b = (a + c) - (a + b)$$
$$= l - L$$

Now
$$I_1 - I = m_a^c - m_a^b$$
$$= m_a(c - b)$$
$$= m_a(l - L)$$

and
$$T_o = m_a(l - L) \cdot \alpha \qquad (9.39)$$

Example 9.14

A connecting rod of an internal combustion engine has a mass of 1.5 kg and the length of the rod is 250 mm. The centre of gravity of the rod is located at a distance of 100 mm from the gudgeon pin. The radius of gyration about an axis through the centre of gravity perpendicular to the plane of rotation is 110 mm. Find the equivalent dynamical system if only one of the masses is located at the gudgeon pin.

If the connecting rod is replaced by two masses, one at the gudgeon pin and the other at the crank pin and the angular acceleration of the rod is 24000 rad/s^2 clockwise, determine the correction couple applied to the system to reduce it to a dynamically equivalent system.

■ **Solution**

Here $m = 1.5$ kg, $L = 250$ mm , $a = 100$ mm, $K = 110$ mm

Now
$$ab = K^2$$
$$b = \frac{110^2}{100} = 121 \text{ mm}$$

Let m_a = mass at the gudgeon pin

m_b = mass at the crank pin

Then
$$m_a = \frac{mb}{L} = \frac{1.5 \times 121}{250} = 0.72 \text{ kg}$$
$$m_b = 1.5 - 0.72 = 0.78 \text{ kg}$$

Correction couple

Now
$$a = 100 \text{ mm}, c = 150 \text{ mm}$$
$$K_1^2 = ac = 100 \times 150 = 15000$$
$$K_1 = 122.47 \text{ mm}$$

Correction couple,
$$T_a = m\left(K_1^2 - K^2\right)\alpha$$
$$= 1.5(15000 - 12100) \times 24000 \times 10^{-6}$$
$$= 104.4 \text{ Nm}$$

Example 9.15

A vertical engine running at 1200 rpm with a stroke of 120 mm, has a connecting rod 300 mm long and of 1.5 kg mass. The mass centre of the rod is 100 mm from the big end centre. When the rod is suspended from the gudgeon pin as a pendulum, it makes 20 complete oscillations in 20 seconds.

(a) Calculate the radius of gyration of the rod about an axis through the mass centre. (b) When the crank is at 35° from the top dead centre and the piston is moving downwards, find the acceleration of the piston and the angular acceleration of the rod. Hence, find the inertia torque exerted on the crankshaft.

■ **Solution**

Angular speed,
$$\omega = \frac{2\pi \times 1200}{60} = 126.66 \text{ rad/s}$$
$$L = 120 \text{ mm} \quad \text{or} \quad r = \frac{L}{2} = 60 \text{ mm}$$
$$l = 300 \text{ mm}, m = 1.5 \text{ kg}, \theta = 35°, n = \frac{l}{r} = \frac{300}{60} = 5$$

(a) Radius of gyration of the rod

Distance of the centre of gravity of rod from the point of suspension,
$$l_1 = 300 - 100 = 200 \text{ mm}$$

Frequency of oscillation of a compound pendulum,
$$f_n = \left(\frac{1}{2\pi}\right)\left[\frac{gl_1}{K^2 + l_1^2}\right]^{0.5}$$
$$\frac{20}{20} = \left(\frac{1}{2\pi}\right)\left[\frac{9.81 \times 200 \times 10^3}{K^2 + 200^2}\right]^{0.5}$$
$$4\pi^2 = \frac{9.81 \times 200 \times 10^3}{K^2 + 200^2}$$

or
$$K^2 + 40000 = 49698$$

or
$$K^2 = 9698$$

or
$$K = 98.47 \text{ mm}$$

(b) Acceleration of the connecting rod:

Angular acceleration of the rod,

$$\alpha_r = -\omega^2 \frac{\sin\theta}{n}$$
$$= -(126.66)^2 \frac{\sin 35°}{5}$$
$$= -1840.35 \text{ rad/s}^2$$

Acceleration of the piston,

$$a_p = \omega^2 r \left[\cos\theta + \frac{\cos 2\theta}{n}\right]$$
$$= (126.66)^2 \times 0.06 \left[\cos 35° + \frac{\cos 70°}{5}\right]$$
$$= 854.33 \text{ m/s}^2$$

Inertia torque exerted on the crankshaft

Mass of the rod at the gudgeon pin,

$$m_g = \frac{m(l - l_1)}{l} = \frac{1.5(300 - 200)}{300} = 0.5 \text{ kg}$$

Vertical inertia force due to m_g,

$$F_i = m_g a_p = 0.5 \times 854.33 = 427.16 \text{ N}$$

Now

$$\sin\phi = \frac{\sin\theta}{n} = \frac{\sin 35°}{5} = 0.1147$$
$$\phi = 6.587°$$
$$\frac{OM}{\sin(\theta + \phi)} = \frac{r}{\cos\phi}$$
$$OM = \frac{60\sin 41.587°}{\cos 6.587°} = 40.1 \text{ mm}$$

Torque due to F_i,

$$T = -F_i \cdot OM = -427.16 \times 0.0401$$
$$= -17.125 \text{ Nm} \quad \text{or} \quad 17.125 \text{ Nm (counter-clockwise)}$$

Equivalent length of a pendulum,

$$l_e = \frac{K^2 + l_1^2}{l_1} = \frac{9698 + 200^2}{200} = 248.49 \text{ mm}$$

Correction couple,

$$T_o = -ml_1(l - l_e)\alpha_r$$
$$= -1.5 \times 200(300 - 248.49)10^{-6} \times 1840.35$$
$$= -28.44 \text{ Nm}$$

Corresponding torque on the crankshaft,

$$T_{cs} = \frac{T_o\cos\theta}{n} = \frac{-28.44\cos 35°}{5} = -4.66 \text{ Nm}$$
$$= 4.66 \text{ Nm (counter-anticlockwise)}$$

Torque due to the mass at the gudgeon pin,

$$T_g = m_g g \cdot OM = 0.5 \times 9.81 \times 0.0401$$
$$= 0.1967 \text{ Nm (clockwise)}$$

Equivalent mass of the rod at the crank pin,

$$m_c = \frac{1.5 \times 200}{300} = 1 \text{ kg}$$

Torque due to this mass,

$$T_c = m_c g \cdot l\sin\phi = 1 \times 9.81 \times 0.3 \times \sin 6.587°$$
$$= 0.3376 \text{ Nm (clockwise)}$$

Inertia torque exerted on the crankshaft $= T + T_s - T_g - T_c$

$= 17.125 + 4.66 - 0.1967 - 0.3376$

$= 21.25$ Nm (counter-clockwise)

Exercises

1 The length of the crank of a reciprocating engine is 120 mm and its connecting rod is 600 mm long. It rotates at 360 rpm and at a particular instant makes an angle of 50° with the inner dead centre. Find the (a) velocity and acceleration of the piston, (b) velocity and acceleration of the midpoint of the connecting rod and (c) angular velocity and angular acceleration of the connecting rod.

2 The following data refers to a steam engine:

Diameter fo the piston = 230 mm
Stroke = 600 mm
Length of connecting rod = 1.5 m
Mass of reciprocating parts = 250 kg
Speed = 120 rpm

Determine the magnitude and direction of the inertia force on the crankshaft when the crank has turned through 30° from inner dead centre.

3 A horizontal steam engine running at 180 rpm has a bore of 320 mm and stroke 560 mm. The connecting rod is 1 m long and the mass of the reciprocating parts is 50 kg. When the crank is 50° past the inner dead centre, the steam pressure on the cover side of the piston is 1.2 MPa, while that on the crank side is 0.15 MPa. Neglecting the area of the piston rod, determine (a) the force on the piston, and (b) the turning moment on the crankshaft.

4 A vertical single cylinder engine has a cylinder diameter of 240 mm and a stroke of 420 mm. The mass of the reciprocating parts is 200 kg. The connecting rod is 4.25 times the crank radius and the speed is 340 rpm. When the crank has turned through 40° from the top dead centre, the net pressure on the piston is 1 MPa. Calculate the effective turning moment on the crankshaft for this position.

5 A petrol engine 100 mm in diameter and 120 mm stroke has a connecting rod 250 mm long. The piston has a mass of 1 kg and the speed is 1800 rpm. The gas pressure is 0.5 MPa at 30° from top dead centre during the explosion stroke. Find (a) the resultant load on the gudgeon pin, (b) the thrust on the cylinder wall, (c) the speed above which the gudgeon pin load will be reversed, and (d) the crank effort at this position.

6 The length of the connecting rod of an engine is 600 mm and its mass is 20 kg. The centre of gravity is 150 mm from the crankpin centre and the crank radius is 120 mm. Determine the dynamically equivalent system keeping one mass at the small end. The frequency of oscillations of the rod when suspended from the centre of the small end is 40 vibrations per minute.

7 A connecting rod 240 mm long has a mass of 2 kg and a moment of inertia of 0.02 kgm^2 about the centre of gravity. The centre of gravity is located at a distance of 150 mm from the small end centre. Determine the dynamically equivalent two-mass system when one mass is located at the small end centre. If the connecting rod is replaced by two-mass system located at the two centers, find the correction couple that must be applied for complete dynamical equivalence of the system when the angular acceleration of the connecting rod is 20 000 rad/s^2 counter-clockwise.

8 A punching machine makes 20 working strokes per minute and is capable of punching 20 mm diameter hole is 15 mm thick steel plate having an ultimate shear strength of 240 MPa. The punching operation takes place during 1/10th of a revolution of the crankshaft. Estimate the power required for the driving motor, assuming a mechanical efficiency of 95%. Determine the size of the rim of the flywheel having width equal to twice the thickness. The flywheel is to revolve 10 times the speed of the crankshaft. The fluctuation of speed is 10%. Assume the flywheel to be made of cast iron having working stress of 6 MPa and density 7300 kg/m^3. The diameter of the flywheel should not exceed 1.5 m Neglect the effect of arms and hub.

9 A vertical double-acting steam engine develops 80 kW at 240 rpm. The maximum fluctuation of energy is 25% of the work done per stroke. The maximum and minimum speeds are not to vary more than $\pm 1\%$ of the mean speed. Find the mass of the flywheel, if the radius of gyration is 0.65 m.

10 The turning moment diagram for a multi-cylinder engine has been drawn to a scale of 1 mm = 5000 N m vertically and 1 mm = 2.5° horizontally. The areas between output torque curve and mean resistance line taken in order from one end are as follows: −340, 21, −245, 300, −118, 230, −225, 37 mm^2, when the engine is running at 180 rpm. If the mass of the flywheel is 1000 kg and the total fluctuation of speed is not to exceed 3% of the mean speed, find the minimum value of the radius of gyration.

11 The torque exerted on the crankshaft of a two stroke engine is given by

$$T = 15000 + 2500 \sin\theta - 2000 \cos 2\theta$$

where θ is the crank angle measured from the inner dead centre. Assuming the resulting torque to be constant, determine (a) the power of the engine when the speed is 180 rpm, (b) the moment of inertia of the flywheel if the speed variation is not to exceed $\pm 0.5\%$ of the mean speed, and (c) the angular acceleration of the flywheel when the crank has turned through 30o from inner dead centre.

12 A steam engine runs at 150 rpm. Its turning moment diagram gave the following area measurements in mm^2 taken in order above and below the mean torque line:

$$500, -250, 270, -390, 190, -340, 270, -250$$

The scale for the turning moment is 1 mm = 500 Nm and for the crank angle is 1 mm = 5°. If the fluctuation of speed is not to exceed $\pm 1.5\%$ of the mean speed, determine the cross-section of the rim of the flywheel assuming it to be rectangular with axial dimension equal to 1.5 times the radial dimension. The hoop stress is limited to 2.5 MPa and the density of the material of the flywheel is 7470 kg/m^3.

13 A cast iron flywheel used for a four stroke internal combustion engine is developing 180 kW at 240 rpm. The hoop stress developed in the flywheel is 5 MPa. The total fluctuation of speed is to be limited to 3% of the mean speed. If the work done during the power stroke is 1/3 times more than the average work done during the whole cycle, find (a) the mean diameter of the flywheel (b) the mass of the flywheel, and (c) the cross-sectional dimensions of the rim when the width is twice the thickness. The density of cast iron may be taken as 7300 kg/m^3.

14 A machine has to carry out punching operations at the rate of 10 holes per minute. It does 6 kNm of work per mm^2 of the sheared area in cutting 25 mm diameter holes in 20 mm thick plates. A flywheel is fitted to the machine shaft which is driven by a constant torque. The fluctuation of speed is between 180 and 200 rpm. The actual punching operation takes 2 seconds. The frictional losses are equivalent to 1/6 th of the work done during punching. Find (a) the power required to drive the punching machine, and (b) the mass of the flywheel, if the radius of gyration of the wheel is 0.5 m.

15 A 17 mm diameter hole is to be punched in a steel plate 19 mm thick at the rate of 20 holes per minute. The actual punching takes place in 1/5th of the interval between punches. The driving motor runs at 1200 rpm and is geared to a countershaft which runs at 160 rpm and upon which the flywheel is mounted. The countershaft in turn is geared to the crankshaft of the press. The resistance to shear for the plate may be taken as 310 MPa. Find (a) the power required for the motor if no flywheel is used, (b) the power required for the motor assuming a flywheel is used, and (c) the mass of the flywheel rim required assuming that 90% of the effective mass at the rim is due to the rim alone. The average speed at the rim diameter is 20 m/s and the coefficient of speed fluctuation is 0.10.

16 The crankshaft of a punching machine runs at a mean speed of 300 rpm. During punching of 10 mm diameter holes in mild steel sheets, the torque required by the machine increases uniformly from 1 kNm to 4 kNm, while the shaft turning through 40°, remains constant for the next 100° and decreases uniformly to 1 kNm for the next 180°. This cycle is repeated during each revolution. The power is supplied by a constant torque motor and the fluctuation of the speed is to be limited to ±3% of the mean speed. Find the power of the motor and the moment of inertia of the flywheel fitted to the machine.

17 Discuss precisely and briefly the fluctuations of a flywheel and a governor with examples of situations demanding the use of one or the other or both on stationary engines.

18 (a) Differentiate between the functions of a flywheel and a governor. In a diesel generating set, is it possible to use only a flywheel or a governor? Give your answer with justifications.

(b) In the design of flywheels, permissible speed variation is an important parameter. State the approximate range of permissible speed variation in percent in case of a diesel engine and a punching machine. Justify your answer.

19 What is an inertia force? Who first introduced the concept of dynamic equilibrium? Explain the concept of dynamic equilibrium for the simple system having two masses m_1 and m_2 attached to the ends of a flexible but inextensible string overhanging a pulley which is free to rotate. Assume that inertia of the pulley and friction on its axle are negligible.

20 A single-cylinder double-acting pump is driven through gearing at 50 rpm. The resisting torque of pump shaft may be assumed to follow a sine curve in half revolution with a maximum value of 6 kNm at 90° and 270°. Find the weight of the flywheel required to be mounted on a pump shaft to keep the speed within 1.5% of the mean speed, if the radius of gyration of the flywheel is 1.5 m. The effect of motor armature and gear wheel is equivalent to a flywheel of 4.5 kN with a radius of gyration of 1 m on the pump shaft.

21 (a) Discuss the effect of inertia force on reciprocating engine mechanism by drawing the free body diagram of each link.

(b) Develop an equation for the relationship between the angular velocities of the input and output cranks of a four-bar linkage.

(c) In analysis of a mechanism, what is a kinematically equivalent system? What conditions should it satisfy to be kinematically equivalent system?

22 (a) The connecting rod of an oil engine weighs 600 N, the distance between the bearing centers is 1 m. The diameter of the big end bearing is 120 mm and of the small end bearing is 75 mm. When suspended vertically with a knife edge through the small end, it makes 100 oscillations in 190 s and with knife edge through the big end it makes 100 oscillations in 165 s. Find the moment of inertia of the rod in kgm^2 and distance of the centre of gravity from the small end centre.

23 500 N body is initially stationary on a 45° incline. The coefficient of dynamic friction between the block and incline is 0.5. What distance along the incline must the weight slide before it attains a speed of 15 m/s?

24 A vertical double-acting steam engine has a cylinder 300 mm diameter and 450 mm stroke and runs at 200 rpm. The reciprocating parts weigh 2250 N and piston rod is 50 mm diameter and the connecting rod is 1.2 m long. When the crank has turned through 125° from the top dead centre, the steam pressure above the piston is 0.3 MPa and below the piston is 0.015 MPa gauge. Calculate the effective turning moment on the crankshaft.

25 A cast iron flywheel is fitted to a punch press to run at 90 rpm and must supply 12 kNm of energy during 1/5th revolution and allow 15% change of speed. The rim speed is limited to 350 m/min. Find the mean diameter and weight of the flywheel and the motor power. Assume overall efficiency as 80%.

26 A machine punching 38 mm diameter holes in a 32 mm thick plate, does 6 Nm of work per square mm of sheared area. The punch has a stroke of 102 mm and punches 6 holes per minute. The maximum speed of the flywheel at its radius of gyration is 27.5 m/s. Find the weight of the flywheel so that its speed at the same radius does not fall below 24.5 m/s. Also, determine the power of the motor driving the machine.

27 Explain how a rigid link be replaced by a dynamically equivalent massless link with two point masses at its ends? Explain how the correction can be applied to the expression for turning moment on crankshaft of a reciprocating engine to account for non-compliance of moment of inertia requirement of dynamic equivalence?

28 What is meant by kinetically equivalent system and what is its application? The connecting rod of a gasoline engine is 300 mm long between its centres. It has a mass of 15 kg and a mass moment of inertia of 7×10^{-3} kgm^2. Its centre of gravity is at 100 mm from its small end centre. Determine the kinetically equivalent two-mass system of the connecting rod if one of the masses is located at small end centre.

29 An engine runs at a constant load at a speed of 480 rpm. The crank effort diagram to a scale of 1 cm = 2 kNm torque and 1 cm = 36° crank angle. The areas of the diagram above and below the mean torque line are measured in square cm and are in the following order:

$$+1.1,\ -1.32,\ +1.53,\ -1.66,\ +1.97,\ -1.62.$$

Design the flywheel if the total fluctuation of speed is not to exceed 10 rpm and the centrifugal stress in the rim is not to exceed 5 N/mm^2. You may assume that the rim breadth is approximately 2.5 times the rim thickness and 90% of the moment of inertia is due to the rim. The density of the material of the flywheel is 7250 kg/m^3.

30 The equation of the turning moment curve of a three crank engine is $(5 + 1.5 \sin 3\theta)$ kNm, where θ is the crank angle. The moment of inertia of the flywheel is 1000 kgm^2 and the mean engine speed is 300 rpm. Calculate (a) the power of the engine, (b) the maximum fluctuation of the speed of the flywheel in percentage, (i) when the resisting torque is constant, and (ii) when the resisting torque is $(5 + 0.6 \sin \theta)$ kNm.

31 A punch press is fitted with a flywheel capable of furnishing 3 kNm of energy during quarter of a revolution near the bottom dead centre while blanking a hole on sheet metal. The maximum speed of the flywheel during the operation is 200 rpm and the speed decreases by 10% during the cutting stroke. The mean radius of the rim is 900 mm. Calculate the approximate weight of the flywheel rim assuming that it contributes 90% of the energy requirements.

32 Diagram of a piston–crank mechanism is shown in Fig.9.18. Crank is rotating in the clockwise direction with an angular velocity ω. Find a relationship between the crank torque T and the piston effort P. Neglect friction. Draw velocity triangle.

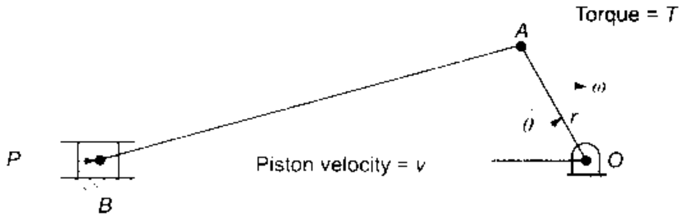


Fig.9.18

33 A punching press is required to punch 40 mm diameter holes in a plate of 15 mm thickness at a rate of 30 holes per minute. It requires 6 N m of energy per mm^2 of sheared area. Determine the moment of inertia of the flywheel if the punching takes one-tenth of a second and the speed of the flywheel varies from 160 rpm to 140 rpm.

34 What is meant by kinetically equivalent system and what is its application? The connecting rod of a gasoline engine is 300 mm long between its centers. It has a mass of 15 kg and a mass moment of inertia of 7×10^{-3} kg m^2. Its centre of gravity is at 100 mm from its small end centre. Determine the kinetically equivalent two-mass system of the connecting rod if one of the masses is located at small end centre.

35 The equation of turning moment for a three-crank engine is:

$$T_e = 25.0 - 7.5 \sin 3\theta \text{ kNm}$$

where θ is the crank angle measured from inner dead centre. The resisting torque exerted by the driven machine is given by:

$$T_r = 25.0 + 3.6 \sin \theta \text{ kNm}$$

The moment of inertia of the flywheel is 360 kg m^2 and the mean engine speed is 450 rpm. Calculate (a) the power of the engine, (b) the maximum fluctuation of flywheel energy per cycle, and (c) the coefficient of fluctuation of speed. [IES,1996]

36 An engine coupled to a machine produces a torque given by $T_e = 10 + \sin 2\theta$ kNm, where θ is the angle of rotation of shaft. The resisting torque of machine is $T_r = 10 + 0.75 \sin \theta$ kNm. The engine runs at a mean speed of 240 rpm and has a flywheel of mass 350 kg and radius of gyration 0.5 m fixed to it. Determine (a) fluctuation of energy, (b) fluctuation of speed and (c) maximum and minimum acceleration of flywheel.

37 The turning moment exerted by two stroke engine at crankshaft is given as:

$$T = 10 + \sin 2\theta - 3 \cos 2\theta \text{ kNm}$$

Where θ is the inclination of crank to inner dead centre. The mass of the flywheel is 600 kg and its radius of gyration 0.8 m. The engine speed is 360 rpm. Assuming external resistance as constant, determine (a) power developed, (b) fluctuation of speed and (c) maximum angular retardation of flywheel.

38 A three-cylinder single acting engine has its cranks set equally at 120° and runs at 750 rpm. The torque-crank angle diagram for each cylinder is a triangle for the power with maximum torque 100 Nm at 60° after dead centre of the corresponding crank. The torque on return stroke is zero. Determine (a) the power developed, (b) the coefficient of fluctuation of speed if the mass of the flywheel is 10 kg and the radius of gyration of 100 mm, (c) the coefficient of fluctuation of energy and (d) maximum angular acceleration of flywheel.

39 A machine is required to punch 4 holes of 40 mm diameter in a plate of 25 m thickness per minute. The work required is 6 Nm per square mm of sheared area. The stroke of punch is 100 mm and maximum speed of flywheel at its radius of gyration is 30 m/s. Find the mass of the flywheel so that the speed does not fall below 27 m/s at the radius of gyration. Also determine the motor power required.

40 A punching press is required to punch 30 mm dia holes in a plate of 20 mm thickness at the rate of 20 holes per minute. It requires 6 Nm of energy per mm^2 of sheared area. If punching takes place 1/10th of a second and the speed of the flywheel varies from 160 to 140 rpm, determine the mass of the flywheel having radius of gyration of 1 m.

41 The following data refers to a horizontal reciprocating engine:

Length of crank = 300 mm
Length of connecting rod = 1.5 m
Speed = 120 rpm, cw
Mass of reciprocating parts of engine = 290 kg
Mass of connecting rod = 250 kg
Centre of gravity of connecting rod from crankpin centre = 475 mm
Radius of gyration of connecting rod about an axis through centre of gravity = 625 mm

Find the inertia torque on the crankshaft when $\theta = 40°$.

42 A single cylinder vertical engine has a bore of 300 mm and a stroke of 400 mm. The connecting rod is 1 m long. The mass of the reciprocating parts is 150 kg. The gas pressure is 0.7 MPa with the crank at 30° from the top dead centre during expansion stroke. The speed of crank is 250 rpm. Determine (a) net force acting on the piston, (b) resultant load on the gudgeon pin, (c) thrust on the cylinder wall, and (d) the speed above which, other things remaining same, the gudgeon pin load would be reversed in direction.